AN ORTHOTROPIC CIRCULAR DISK SUBJECTED TO ITS OWN WEIGHT WHEN SUPPORTED AT A POINT

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TABLE OF CONTENTS

TABLE OF CONTENTS (CONTD.)

LIST OF FIGURES

NOTATION

(The symbols are given for the x- and y-directions, x, y, and z being taken as rectangular coordinates; if the z-direction is to be considered, suitable changes in subscripts and definitions are easily made.)

σ_x, σ_y	Normal components of stress parallel to the x- and y-axes.
τ_{xy}	Shearing stress component parallel to the y-axis.
ϵ_x, ϵ_y	Strain components in the x- and y-directions.
γ_{xy}	Shearing strain component in a plane parallel to the xy-plane.
E_x, E_y	Moduli of elasticity (Young's moduli) for the x- and y-directions.
G_{xy}	Modulus of elasticity in shear (modulus of rigidity) associated with the directions of x and y.
ν_{xy}, ν_{yx}	Poisson's ratio associated with the directions of x and y; ν_{xy} is the ratio of the contraction in the y-direction to the extension in the x-direction caused by a tensile force in the x-direction, and ν_{yx} is defined similarly.
u, v	Components of displacement in the x- and y-directions.

INTRODUCTION

A substance for which the elastic properties are independent of direction is said to be isotropic. If the substance is not isotropic but possesses three mutually perpendicular planes of elastic symmetry, it is called orthotropic. A plate of uniform thickness made of orthotropic material in such a way that the faces of the plate are parallel to a plane of elastic symmetry will have two perpendicular axes of elastic symmetry in its own plane and is called an orthotropic plate. The objective of this work is the determination of the stress distribution in a circular disk, i.e., a circular plate, of orthotropic material subjected to its own weight when supported, with its plane vertical, (1) at a point on its boundary and (2) at a point at its center. Point support, as considered here, implies uniform support along a line segment joining the faces of the disk and perpendicular to each face at the point specified.

As a practical example, a board cut from a log in such a way that a transverse cross section is on a diameter of the log (quarter-sawn board), or in such a way that, at its midpoint, a transverse cross section is perpendicular to a diameter of the log (plain-sawn board), may be considered as an orthotropic plate, and a disk taken from such a board may be considered as an orthotropic disk. The problem considered here, then, is the mathematical idealization of the

problem of a wooden disk suspended in a vertical plane from a point on its boundary or, in the second case, from a slender peg at its center.

The stress distribution in a circular isotropic disk subjected to its own weight when supported, with its plane vertical, at any point has been obtained by R. D. Mindlin.[1] The problem of the stress distribution in a circular orthotropic disk under various edge loadings has been solved by H. F. Cleaves,[2] using a complex potential function, but the weight of the disk was not considered.

This discussion is divided into three chapters. Chapter I furnishes, in a sense, the tools for the investigation, containing a derivation of the differential equation governing the stress distribution in an orthotropic plate in a state of plane stress, a discussion of a general solution of this equation, and a consideration of the effect of boundary forces. The stress distribution in the disk supported at a boundary point is obtained in Chapter II, and the problem of the disk supported at its center is solved in Chapter III. In these treatments, the stresses are

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[1] R. D. Mindlin, "Stress in a Heavy Disk Suspended from an Eccentric Peg," Journal of Applied Physics, Vol. 9, Nov., 1938, pp. 714-717.

[2] H. F. Cleaves, "The Stresses in an Aeolotropic Circular Disk," Quarterly Journal of Mechanics and Applied Mathematics, Vol. VIII, Part 1, March, 1955, pp. 59-80.

obtained from a stress function, the form of which is assumed and the constants of which are evaluated from boundary conditions and from considerations of continuity and single-valuedness. In each of Chapters II and III the displacements in the disk are obtained following the determination of the stresses, and in Chapter II the stress function for the orthotropic disk is shown to contain as a limiting case the stress function for an isotropic disk under the same conditions. Differences in coordinate systems and method of attack make a comparison of this isotropic stress function with that obtained by Mindlin unprofitable, but direct solution by the author of the problem for the isotropic disk has verified that the limiting case referred to above is, in fact, the isotropic stress function.

CHAPTER I

EQUATIONS GOVERNING THE STRESS DISTRIBUTION IN AN ORTHOTROPIC PLATE SUBJECTED TO ITS OWN WEIGHT UNDER THE CONDITION OF PLANE STRESS.

1.1. Derivation of the Differential Equation Governing Stress Distribution in an Orthotropic Plate Under Plane Stress.

Let the plate be referred to a space rectangular coordinate system (Fig. 1) with the axes taken parallel to the planes of elastic symmetry, and let the displacements in the directions of the x-, y-, and z-axes be $u = u(x,y,z)$, $v = v(x,y,z)$, and $w = w(x,y,z)$, respectively. Then the

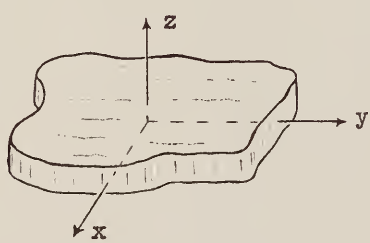

Fig. 1

components of stress and strain are related as follows:[3]

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[3] H. W. March, Stress-Strain Relations in Wood and Plywood Considered as Orthotropic Materials, Forest Products Laboratory Report No. R1503, February, 1944, p. 2.

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = \frac{1}{E_x}\sigma_x - \frac{\nu_{yx}}{E_y}\sigma_y - \frac{\nu_{zx}}{E_z}\sigma_z \;, \\
\epsilon_y &= \frac{\partial v}{\partial y} = -\frac{\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y - \frac{\nu_{zy}}{E_z}\sigma_z \;, \\
\epsilon_z &= \frac{\partial w}{\partial z} = -\frac{\nu_{xz}}{E_x}\sigma_x - \frac{\nu_{yz}}{E_y}\sigma_y + \frac{1}{E_z}\sigma_z \;, \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{G_{xy}}\tau_{xy} \;, \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{1}{G_{yz}}\tau_{yz} \;, \\
\gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{1}{G_{zx}}\tau_{zx} \;.
\end{aligned} \tag{1}$$

In these equations E_x, E_y, and E_z are Young's moduli in the x-, y-, and z-directions, respectively, and G_{xy}, G_{yz}, and G_{zx} are the moduli of rigidity associated with the directions of x and y, y and z, and z and x, respectively. The symbol ν represents Poisson's ratio; in particular, ν_{xy} is the ratio of the contraction in the y-direction to the extension in the x-direction caused by a tensile force in the x-direction.

If the plate is loaded only by forces parallel to the plane of the plate and distributed uniformly over the thickness of the plate, then the stress components σ_z, τ_{yz}, and τ_{zx} are zero on the faces of the plate and, for a thin plate, it may be assumed that these components are zero within the

plate, also. The remaining stress components, σ_x, σ_y, and τ_{xy}, may then be assumed to be independent of z, that is, assumed to be functions of x and y only. Such a state of stress is called plane stress, and will be taken to represent the stress condition in the treatments of this and the following chapters.

Under the condition of plane stress, equations (1) become

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E_x}\sigma_x - \frac{\nu_{yx}}{E_y}\sigma_y \quad ,$$

$$\epsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y \quad , \tag{2}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{G_{xy}}\tau_{xy} \quad .$$

The equations of equilibrium for plane stress,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \quad . \tag{3}$$

in which X and Y are the x- and y-components, respectively, of the body force per unit volume, are obtained by applying the conditions of static equilibrium to an elementary block of the plate, shown with its applied stresses in Fig. 2. If the body force acting on the plate is that due to gravity, and if the plate be so oriented that the x-axis is downward,

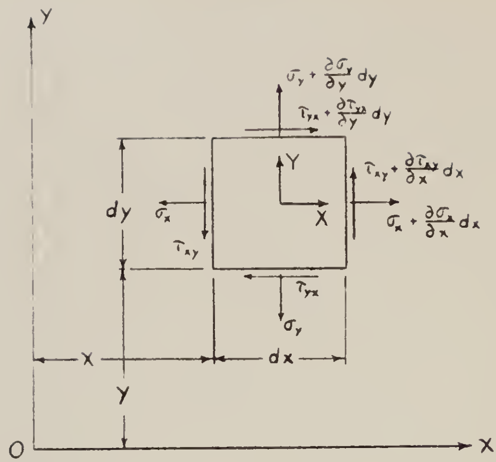

Fig. 2

then

$$X = \rho g \quad \text{and} \quad Y = 0, \tag{4}$$

where ρ is the mass density of the material and g is the acceleration due to gravity. For this situation, equations (3) become

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0 \ . \end{aligned} \tag{5}$$

These equations will be satisfied if there is introduced a stress function F(x,y) [4] such that

[4] G. B. Airy, Report of the British Association for the Advancement of Science, 1862.

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} - \rho g x, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} - \rho g x, \quad \tau_{xy} = - \frac{\partial^2 F}{\partial x \partial y} \quad . \tag{6}$$

The requirement that the displacements u(x,y) and v(x,y) be single-valued and continuous (class three) gives rise, in the case of plane stress, to the following differential equation in the strains, known as the compatibility equation for plane stress:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad . \tag{7}$$

Equations (6) substituted into equations (2) yield

$$\begin{aligned} \epsilon_x &= \frac{1}{E_x}\left(\frac{\partial^2 F}{\partial y^2} - \rho g x\right) - \frac{\nu_{yx}}{E_y}\left(\frac{\partial^2 F}{\partial x^2} - \rho g x\right) \quad , \\ \epsilon_y &= - \frac{\nu_{xy}}{E_x}\left(\frac{\partial^2 F}{\partial y^2} - \rho g x\right) + \frac{1}{E_y}\left(\frac{\partial^2 F}{\partial x^2} - \rho g x\right) \quad , \\ \gamma_{xy} &= - \frac{1}{G_{xy}} \frac{\partial^2 F}{\partial x \partial y} \quad . \end{aligned} \tag{8}$$

Using these expressions for the strains in equation (7) and using the relation[5]

$$\frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x} \quad , \tag{9}$$

we obtain the differential equation

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[5] March, op. cit., p. 7.

$$\frac{1}{E_y}\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_{xy}} - \frac{2\,\nu_{xy}}{E_x}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x}\frac{\partial^4 F}{\partial y^4} = 0 \quad . \tag{10}$$

Under the substitution[6]

$$\eta = \epsilon y \quad , \tag{11}$$

with

$$\epsilon = \sqrt[4]{\frac{E_x}{E_y}} \quad , \tag{12}$$

equation 10 ~~(9)~~ becomes, upon multiplication by $E_x E_y$,

$$\frac{\partial^4 F}{\partial x^4} + 2K\frac{\partial^4 F}{\partial x^2 \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0 \quad , \tag{13}$$

where

$$K = \frac{\sqrt{E_x E_y}}{2}\left(\frac{1}{G_{xy}} - \frac{2\,\nu_{xy}}{E_x}\right) \quad . \tag{14}$$

From the foregoing derivation of equation (13), it is clear that any function F which satisfies that equation will, by means of relations (6), satisfy the equilibrium equations (5) and the compatibility equation (7). Some information on the form of the stress function is given in the section which follows.

1.2. A General Solution of the Differential Equation.

Assuming a solution of equation (13) to have the form

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[6] H. W. March, Flat Plates of Plywood Under Uniform or Concentrated Loads, Forest Laboratory Report No. 1312, March 1942, p. 40.

$F = F(x + \mu\eta)$, we are led to the auxiliary equation

$$\mu^4 + 2K\mu^2 + 1 = 0 \quad , \tag{15}$$

a quadratic equation in μ^2 with roots

$$\mu^2 = -K \pm \sqrt{K^2 - 1} \quad . \tag{16}$$

For wood, K, as defined by equation (14), is probably always greater than 1.[7]

Let

$$K = \cosh\varphi \quad . \tag{17}$$

Then

$$\begin{aligned} \mu^2 &= -\cosh\varphi \pm \sqrt{\cosh^2\varphi - 1} \\ &= -\cosh\varphi \pm \sinh\varphi \\ &= -\frac{e^{\varphi} + e^{-\varphi}}{2} \pm \frac{e^{\varphi} - e^{-\varphi}}{2} \\ &= -e^{\pm\varphi} \quad . \end{aligned} \tag{18}$$

Hence, the roots of equation ~~(14)~~ 15 may be written $\mu = \pm ie^{\pm\frac{\varphi}{2}}$, or

$$\mu_1 = i\alpha,\ \mu_2 = -i\alpha,\ \mu_3 = i\beta,\ \mu_4 = -i\beta, \tag{19}$$

where

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[7] C. B. Smith, Effect of Elliptic or Circular Holes on the Stress Distribution in Plates of Wood or Plywood Considered as Orthotropic Materials, Forest Products Laboratory Report No. 1510, 1944, p. 5.

$$\alpha = e^{\frac{\varphi}{2}} = \sqrt{K + \sqrt{K^2 - 1}}$$

and (20)

$$\beta = e^{-\frac{\varphi}{2}} = \sqrt{K - \sqrt{K^2 - 1}} \quad .$$

It will be assumed throughout this work that $\alpha \neq \beta$, unless specifically stated otherwise.

A general solution of equation (13) is, then,

$$F = F_1(x+i\alpha\eta) + F_2(x-i\alpha\eta) + F_3(x+i\beta\eta) + F_4(x-i\beta\eta) \quad , \tag{21}$$

where the functions F_j, $j = 1, 2, 3, 4$, are any analytic functions of their respective arguments.

The requirement that the stresses obtained from this stress function be real, leads one to consider the real part of F, denoted by $R\{F\}$, as a possible solution of equation (13). That $R\{F\}$ is, indeed, a solution of that equation can be seen by examining separately the terms $R\{F_j\}$, $j = 1, 2, 3, 4$, of $R\{F\}$. For example, substitution of $R\{F_1(x + i\alpha\eta)\}$ for F in equation (13) gives

$$(1 - 2K\alpha^2 + \alpha^4)R\,\frac{d^4F_1(x + i\alpha\eta)}{d(x + i\alpha\eta)^4} = 0 \quad . \tag{22}$$

This equation is satisfied by virtue of equation (17) and the first of equations (19) and (20). Hence, $R\{F_1(x + i\alpha\eta)\}$ is a solution of equation (13). Similar results obtain for $R\{F_2(x - i\alpha\eta)\}$, $R\{F_3(x + i\beta\eta)\}$, and $R\{F_4(x - i\beta\eta)\}$, whence it follows that $R\{F\}$, with F given by equation(21), is a solution of equation (13). And it is $R\{F\}$, then, which will be

used in place of F in equations (6) to give the (real) stresses.

Since, for $f(z)$ analytic, the real part of $f(\bar{z})$ is harmonic, we can obtain an analytic function $g(z)$ such that $R\{g(z)\} = R\{f(\bar{z})\}$. Thus, the function $R\{F\}$, from equation (21), may be written

$$R\{F\} = R\{G_1(x + i\alpha\eta) + G_2(x + i\beta\eta)\} \quad , \tag{23}$$

where G_1 and G_2 are analytic functions of their respective arguments.

The problem of obtaining the stress distribution in such a plate as is considered here is solved, then, when there is found a function in the form of equation (23) such that the corresponding stresses and strains satisfy the boundary conditions on the plate.

It is apparent from equation (23) that it will be useful to introduce two new systems of orthogonal Cartesian coordinates (x_1, y_1) and (x_2, y_2) related to the coordinates (x,η) by the formulas

$$x_1 + iy_1 = z_1 = x + i\alpha\eta \text{ and } x_2 + iy_2 = z_2 = x + i\beta\eta. \tag{24}$$

Hence, to any point P in the region R lying in the z-plane correspond points P_1 and P_2 inside some regions R_1 and R_2 which lie in the complex z_1- and z_2-planes, respectively. In order to solve a problem of the proposed type, we shall make use of two mapping functions $z_1 = z_1(\zeta_1)$ and $z_2 = z_2(\zeta_2)$

which are analytic functions of the complex variables ζ_1 and ζ_2, respectively, and which map the regions R_1 and R_2 into regions more easily discussed.

1.3. Effect of Boundary Forces.

If the force on the boundary per unit length and unit thickness has x- and y-components denoted by X_ν and Y_ν, respectively, then the following relations must be satisfied on the boundary:[8]

$$
\begin{aligned}
X_\nu &= \sigma_x \frac{dy}{ds} - \tau_{xy} \frac{dx}{ds} \ , \\
Y_\nu &= \tau_{xy} \frac{dy}{ds} - \sigma_y \frac{dx}{ds} \ ,
\end{aligned}
\tag{25}
$$

where ds denotes an element of the boundary. With the stresses given by equations (6), equations (25) may be rewritten in terms of the stress function to yield

$$
\begin{aligned}
X_\nu &= \frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) - \rho g x \frac{dy}{ds} \ , \\
Y_\nu &= -\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) + \rho g x \frac{dx}{ds} \ .
\end{aligned}
\tag{26}
$$

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[8] S. Timoshenko and J. N. Goodier, Theory of Elasticity, McGraw Hill Co., New York, 1951, p. 190.

CHAPTER II

STRESS DISTRIBUTION IN AN ORTHOTROPIC DISK SUBJECTED TO ITS OWN WEIGHT WHEN SUPPORTED AT A POINT ON ITS BOUNDARY.

2.1. Boundary Conditions.

Consider an orthotropic disk of radius a and, for convenience, of unit thickness. Let the center of the disk be taken as the origin of a plane rectangular coordinate system (Fig. 3) with x- and y-axes parallel to the axes of elastic symmetry. The disk will be considered to be so oriented that the force on it due to gravity has the same direction as the positive x-axis. The case of point support (at the point (a, 0)) will be considered as the limit of the case in which the supporting force $\rho g \pi a^2$ is distributed over

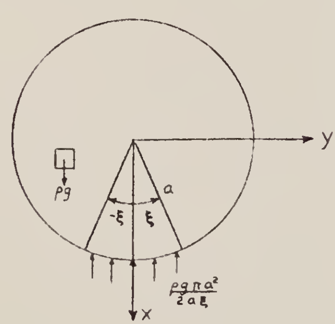


Fig. 3

an arbitrarily small arc of the boundary, according to the following boundary conditions (see Fig. 3):

$$X_\nu = -\frac{\rho g \pi a^2}{2a\xi}\ , \text{ for } -\xi \leqq \theta \leqq \xi\ ;$$

$$X_\nu = 0\ , \text{ for all other values of } \theta;\qquad (27)$$

$$Y_\nu = 0\ , \text{ for all values of } \theta.$$

In these conditions, θ is an angle measured from the positive x-axis, and ξ is an arbitrarily small (positive) angle.

2.2. Mapping Functions.

Referring to the discussion of mappings on page 12, we observe that the regions R_1 and R_2 are in this case bounded by elliptical curves given, respectively, by the equations

$$z_1 = a(\cos\theta + i\alpha\epsilon\sin\theta)$$

and (28)

$$z_2 = a(\cos\theta + i\beta\epsilon\sin\theta)\ .$$

Let ζ_1 and ζ_2 be functions of z_1 and z_2, respectively, which map these boundaries into the unit circles $\zeta_1 = e^{i\theta}$ and $\zeta_2 = e^{i\theta}$ in the ζ_1- and ζ_2-planes. Then, in terms of ζ_1,

$$\cos\theta = \frac{1}{2}(\zeta_1 + \frac{1}{\zeta_1}) \text{ and } \sin\theta = \frac{1}{2i}(\zeta_1 - \frac{1}{\zeta_1})\quad ,\qquad (29)$$

so that

$$z_1 = \frac{a}{2}(\zeta_1 + \frac{1}{\zeta_1}) + \frac{a\alpha\epsilon}{2}(\zeta_1 - \frac{1}{\zeta_1})\quad ,$$

or

$$z_1 = \frac{a(1+\alpha\epsilon)}{2}\zeta_1 + \frac{a(1-\alpha\epsilon)}{2}\frac{1}{\zeta_1}\ .\qquad (30)$$

Similarly, considering ς_2 and z_2, it is found that

$$z_2 = \frac{a(1+\beta\epsilon)}{2}\varsigma_2 + \frac{a(1-\beta\epsilon)}{2}\frac{1}{\varsigma_2} \quad . \tag{31}$$

Solution of equation (30) for ς_1 and equation (31) for ς_2 yields

$$\varsigma_1 = \frac{1}{a(1+\alpha\epsilon)}[z_1 \pm \sqrt{z_1^2 + a^2(\alpha^2\epsilon^2 - 1)}]$$

and (32)

$$\varsigma_2 = \frac{1}{a(1+\beta\epsilon)}[z_2 \pm \sqrt{z_2^2 + a^2(\beta^2\epsilon^2 - 1)}] \quad .$$

If we let

$$r_1 = \alpha\epsilon \quad , \qquad r_2 = \beta\epsilon \quad , \tag{33}$$

$$W_1 = \pm\sqrt{z_1^2 + a^2(r_1^2 - 1)} \text{ and } W_2 = \pm\sqrt{z_2^2 + a^2(r_2^2 - 1)} \tag{34}$$

then

$$\varsigma_1 = \frac{z_1 + W_1}{a(1 + r_1)}$$

and (35)

$$\varsigma_2 = \frac{z_2 + W_2}{a(1 + r_2)} \quad .$$

The function W_1 has branch points $z_1 = \pm a\sqrt{1 - r_1^2}$, which are the foci of the first ellipse (28). Let the line segment joining these points be taken as a branch cut. From equations (12), (20), and (33), it may be seen that $r_1 > 0$ and $r_2 > 0$. If $r_1 < 1$, then, on the branch cut

$$z_1 = x, \text{ with } |x| \leq a\sqrt{1 - r_1^2} \quad , \tag{36}$$

and

$$\zeta_1 = \frac{1}{a(1 + r_1)}[x \pm \sqrt{x^2 + a^2(r_1^2 - 1)}\ ]$$

$$= \frac{x}{a(1 + r_1)} \pm i\,\frac{\sqrt{a^2(1 - r_1^2) - x^2}}{a(1 + r_1)} \quad . \tag{37}$$

Representing ζ_1 by $\chi + i\psi$, with χ and ψ real, we have

$$\chi = \frac{x}{a(1 + r_1)} \quad \text{and} \quad \psi = \pm\frac{\sqrt{a^2(1 - r_1^2) - x^2}}{a(1 + r_1)} \quad , \tag{38}$$

whence

$$\chi^2 + \psi^2 = \frac{1 - r_1}{1 + r_1} \quad . \tag{39}$$

Thus, the image in the ζ_1-plane of the branch cut in the z_1-plane is a circle of radius $\sqrt{\frac{1 - r_1}{1 + r_1}}$. If $r_1 > 1$, then the branch points are on the axis of imaginaries in the z_1-plane, and, on the branch cut,

$$z_1 = iy, \text{ with } |y| \leqq a\sqrt{r_1^2 - 1} \ . \tag{40}$$

An analysis like the one above shows, however, that the image in the ζ_1-plane is again a circle, this time with radius $\sqrt{\frac{r_1 - 1}{r_1 + 1}}$. Let

$$c = \sqrt{\frac{r_1 - 1}{r_1 + 1}} \quad . \tag{41}$$

In either case, then, the function ζ_1 maps the region R_1 bounded by the first ellipse (28), with the cut joining its foci, into the region in the ζ_1-plane lying between the concentric circles $|\zeta_1| = |c|$ and $|\zeta_1| = 1$. It can be shown

similarly that the function ζ_2 maps the region R_2 bounded by the second ellipse (28), with a branch cut joining its foci, into the region in the ζ_2-plane lying between the concentric circles $|\zeta_2| = |d|$ and $|\zeta_2| = 1$, where

$$d = \sqrt{\frac{r_2 - 1}{r_2 + 1}} \quad . \tag{42}$$

2.3. <u>The Stress Function; Conditions for Single-Valuedness and Continuity.</u>

The stress distribution in the disk under consideration is obtained by finding a solution of equation (13) in the form of equation (23), i.e., the real part of the sum of an analytic function of z_1 and an analytic function of z_2, which satisfies equations (26) on the boundary of the disk, with X_ν and Y_ν given by equations (27). The ring-shaped regions into which the regions R_1 and R_2 of section 2.2, with their branch cuts, are mapped by the functions ζ_1 and ζ_2 suggest the use of Laurent series in ζ_1 and ζ_2 for this purpose; in particular, they suggest the assumptions

$$\frac{\partial F}{\partial x} = R\{\sum_{-\infty}^{\infty} a_n \zeta_1^n + \sum_{-\infty}^{\infty} b_n \zeta_2^n\}$$

and (43)

$$\frac{\partial F}{\partial y} = R\{\sum_{-\infty}^{\infty} i r_1 a_n \zeta_1^n + \sum_{-\infty}^{\infty} i r_2 b_n \zeta_2^n\} \quad ,$$

where a_n and b_n are complex coefficients to be determined by the boundary conditions.

The requirement that the stresses obtained from this

stress function F be single-valued is tantamount to the requirement that ζ_1 and ζ_2 be single-valued functions, i.e., that the variables z_1 and z_2 be not allowed to cross the branch cuts in their planes. This, in turn, amounts to the stipulation that the signs taken with W_1 and W_2 (equations (34)) be such that

$$|\zeta_1| = \left|\frac{z_1 + W_1}{a(1 + r_1)}\right| \geqq \left|\sqrt{\frac{r_1 - 1}{r_1 + 1}}\right|$$

and (44)

$$|\zeta_2| = \left|\frac{z_2 + W_2}{a(1 + r_2)}\right| \geqq \left|\sqrt{\frac{r_2 - 1}{r_2 + 1}}\right| ,$$

i.e., such that

$$|z_1 + W_1| \geqq a\left|\sqrt{r_1^2 - 1}\right|$$

and (45)

$$|z_2 + W_2| \geqq a\left|\sqrt{r_2^2 - 1}\right| .$$

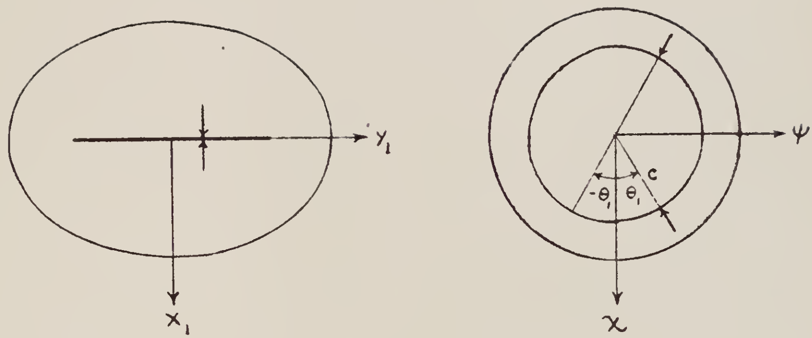
A further consideration, if the stresses are to be continuous, is the behavior of $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in the vicinity of the branch cuts in the z_1- and z_2-planes. Assume for the moment that $r_1 > 1$, so that the branch cut in the z_1-plane is on the imaginary axis. As a point on the cut is approached from below (see Fig. 4),

$$\zeta_1 \longrightarrow ce^{i\theta_1} , \tag{46}$$

and as the same point is approached from above,

$$\zeta_1 \longrightarrow - ce^{-i\theta_1} , \tag{47}$$

where θ_1 depends on the particular point approached. The



z_1-plane $\qquad\qquad$ ζ_1-plane

Fig. 4

values approached by $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ as ζ_1 approaches its limit must be the same for both cases. It is seen upon reference to equations (43) that this requirement will be met if

$$\sum_{-\infty}^{\infty} a_n c^n e^{ni\theta_1} = \sum_{-\infty}^{\infty} a_n (-c)^n e^{-ni\theta_1} \quad . \tag{48}$$

Equating the coefficients of like powers of $e^{i\theta_1}$, we are led to the relations

$$a_{-n} = (-c^2)^n a_n \ , \ n = 0, \pm 1, \pm 2, \ldots \ . \tag{49}$$

If $r_2 > 1$, this same process, with subscripts 1 replaced by subscripts 2 and c replaced by d, leads to the relation

$$b_{-n} = (-d^2)^n b_n \ , \ n = 0, \pm 1, \pm 2, \ldots \ . \tag{50}$$

If one of the constants r_1 and r_2 is less than 1, then the corresponding branch cut is on the real axis. The case $r_1 < 1$, for example, implies, by equation (41), that c is

imaginary, so that $|c| = -ic$. As z_1 approaches a point on the branch cut from the right,

$$\zeta_1 \longrightarrow (-ic)e^{i\theta_1} \quad , \tag{51}$$

and as z_1 approaches the same point from the left,

$$\zeta_1 \longrightarrow (-ic)e^{-i\theta_1} \quad . \tag{52}$$

Now, the requirement that the values approached by $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ be the same for both cases can be shown to lead again to relation (49). If $r_2 < 1$, this process, with subscripts 1 replaced by subscripts 2 and c replaced by d, yields relation (50) again.

2.4. <u>The Constants Determined from the Boundary Conditions</u>.

For the boundary of the disk, where $\zeta_1 = \zeta_2 = e^{i\theta}$, equations (43) may be written

$$\frac{\partial F}{\partial x} = R\{\sum_{-\infty}^{\infty} (a_n + b_n)e^{ni\theta}\}$$

and (53)

$$\frac{\partial F}{\partial y} = R\{i \sum_{-\infty}^{\infty} (r_1 a_n + r_2 b_n)e^{ni\theta}\} \quad .$$

Now, $x = a \cos \theta$ and $y = a \sin \theta$ on the boundary, so that

$$\frac{d\theta}{ds} = \frac{1}{a} \tag{54}$$

and

$$\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) = R\{\sum_{-\infty}^{\infty}{}^{*} ni(a_n + b_n)e^{ni\theta} \cdot \frac{1}{a}\}$$

and (55)

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) = R\{\sum_{-\infty}^{\infty}{}^{*} - n(r_1 a_n + r_2 b_n)e^{ni\theta} \cdot \frac{1}{a}\} \quad ,$$

where the symbol Σ^* is used to indicate that the constant term, corresponding to $n = 0$, is missing.

Substituting these expressions for $\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right)$ and $\frac{d}{ds}\left(\frac{\partial F}{\partial y}\right)$ in equations (26), and noting that

$$\rho g x \frac{dy}{ds} = \rho g a \cos\theta \cdot a\cos\theta \cdot \frac{1}{a} = \rho g a \cos^2\theta$$

$$= \frac{\rho g a}{2}(1 + \cos 2\theta) = \frac{\rho g a}{4}(2 + e^{2i\theta} + e^{-2i\theta})$$

and

$$\rho g x \frac{dx}{ds} = \rho g a \cos\theta \cdot (-a\sin\theta)\frac{1}{a} = -\rho g a \sin\theta\cos\theta \tag{56}$$

$$= -\frac{\rho g a}{2}\sin 2\theta = \frac{\rho g a i}{4}(e^{2i\theta} - e^{-2i\theta}) \quad ,$$

we obtain

$$X_\nu = -\frac{1}{a}R\{\sum_{-\infty}^{\infty}{}^* n(r_1 a_n + r_2 b_n)e^{ni\theta}\} - \frac{\rho g a}{4}(2 + e^{2i\theta} + e^{-2i\theta})$$

and (57)

$$Y_\nu = -\frac{1}{a}R\{i\sum_{-\infty}^{\infty}{}^* n(a_n + b_n)e^{ni\theta}\} + \frac{\rho g a i}{4}(e^{2i\theta} - e^{-2i\theta}) \quad .$$

The identity $R\{w\} = \frac{1}{2}(w + \bar{w})$, where $\bar{w}$ denotes the complex conjugate of w, permits the rewriting of equations (57) in the form

$$X_\nu = -\frac{1}{2a}\sum_{-\infty}^{\infty}{}^*[n(r_1 a_n + r_2 b_n)e^{ni\theta} + n(r_1\bar{a}_n + r_2\bar{b}_n)e^{-ni\theta}]$$

$$- \frac{\rho g a}{4}(2 + e^{2i\theta} + e^{-2i\theta})$$

and (58)

$$Y_\nu = -\frac{1}{2a}\sum_{-\infty}^{\infty}{}^*[n(a_n + b_n)e^{ni\theta} - n(\bar{a}_n + \bar{b}_n)e^{-ni\theta}]$$

$$+ \frac{\rho g a i}{4}(e^{2i\theta} - e^{-2i\theta}) \quad .$$

Let the function X_ν given by the first two of equations

(27) be represented by a complex Fourier series as follows:

$$X_\nu = \sum_{-\infty}^{\infty} c_n e^{ni\theta} \quad , \tag{59}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_\nu e^{-ni\theta} d\theta = \frac{1}{2\pi}\int_{-\xi}^{\xi}\left(-\frac{\rho g\pi a}{2\xi}\right)e^{-ni\theta}d\theta \quad . \tag{60}$$

Integrating, we obtain

$$\begin{aligned} c_n &= \frac{1}{\pi}\left(-\frac{\rho g\pi a}{2\xi}\right)\int_0^{\xi} e^{-ni\theta}d\theta \\ &= -\frac{\rho g a}{2\xi}\,\frac{e^{-ni\xi}-1}{-ni} \\ &= -\frac{\rho g a}{2}\,\frac{e^{-ni\xi}-1}{-ni\xi} \quad , \\ & \qquad n = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{61}$$

These constants represent the only appearance of the quantity ξ in this discussion. Their limits as ξ approaches zero, representing the desired case of point support of the disk, are found by l'Hospital's rule to be

$$c_n = -\frac{\rho g a}{2} \quad , \quad n = 0, \pm 1, \pm 2, \ldots \tag{62}$$

Hence, by equation (59),

$$X_\nu = -\frac{\rho g a}{2}\sum_{-\infty}^{\infty} e^{ni\theta} \quad . \tag{63}$$

The third of equations (27) states that

$$Y_\nu = 0 \quad , \text{ for all values of } \theta.$$

With the use of these last two equations, equations (58) become

$$- \frac{1}{2a} \sum_{-\infty}^{\infty}{}^{*} \left[n(r_1 a_n + r_2 b_n) e^{ni\theta} + n(r_1 \bar{a}_n + r_2 \bar{b}_n) e^{-ni\theta} \right]$$

$$- \frac{\rho g a}{4}(2 + e^{2i\theta} + e^{-2i\theta}) = - \frac{\rho g a}{2} \sum_{-\infty}^{\infty} e^{ni\theta}$$

and (64)

$$- \frac{1}{2a} \sum_{-\infty}^{\infty}{}^{*} \left[n(a_n + b_n) e^{ni\theta} - n(\bar{a}_n + \bar{b}_n) e^{-ni\theta} \right]$$

$$+ \frac{\rho g a i}{4}(e^{2i\theta} - e^{-2i\theta}) = 0 \quad .$$

In each of these equations the coefficients of like powers of $e^{i\theta}$ in the lefthand and righthand members must be equal, in order that the equations may hold for all values of θ. Equating the coefficients of $e^{ni\theta}$ in the two members of each of equations (64), we obtain

$$- \frac{1}{2a}[n(r_1 a_n + r_2 b_n)] + \frac{1}{2a}[n(r_1 \bar{a}_{-n} + r_2 \bar{b}_{-n})] = - \frac{\rho g a}{2}$$

and (65)

$$- \frac{1}{2a}[n(a_n + b_n)] - \frac{1}{2a}[n(\bar{a}_{-n} + \bar{b}_{-n})] = 0 \quad ,$$

$$n = \pm 1, \pm 3, \pm 4, \ . \ . \ . \ ,$$

and, for $n = \pm 2$,

$$- \frac{1}{2a}[n(r_1 a_n + r_2 b_n)] + \frac{1}{2a}[n(r_1 \bar{a}_{-n} + r_2 \bar{b}_{-n})] - \frac{\rho g a}{4} = - \frac{\rho g a}{2}$$

and (66)

$$- \frac{1}{2a}[n(a_n + b_n)] - \frac{1}{2a}[n(\bar{a}_{-n} + \bar{b}_{-n})] \pm \frac{\rho g a}{4} = 0 \quad ,$$

where the sign of the term $\frac{\rho g a}{4}$ in the last equation is the same as that of n. In the expressions for $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ given by equations (43), the terms involving a_o and b_o are constant

terms and hence do not affect the stresses, given by equations (6). There is no need, therefore, to consider the case $n = 0$ in evaluating a_n and b_n.

Equations (65) are equivalent to the following system of equations, wherein n may assume the values 1, 3, 4, . . . :

$$
\begin{aligned}
r_1 a_n + r_2 b_n - r_1 \bar{a}_{-n} - r_2 \bar{b}_{-n} &= \frac{\rho g a^2}{n} , \\
- r_1 a_{-n} - r_2 b_{-n} + r_1 \bar{a}_n + r_2 \bar{b}_n &= \frac{\rho g a^2}{n} , \\
a_n + b_n + \bar{a}_{-n} + \bar{b}_{-n} &= 0 ,
\end{aligned}
\tag{67}
$$

and

$$
a_{-n} + b_{-n} + \bar{a}_n + \bar{b}_n = 0 .
$$

With the use of equations (49) and (50), this system may be written

$$
\begin{aligned}
r_1 a_n + r_2 b_n - r_1(-c^2)^n \bar{a}_n - r_2(-d^2)^n \bar{b}_n &= \frac{\rho g a^2}{n} , \\
- r_1(-c^2)^n a_n - r_2(-d^2)^n b_n + r_1 \bar{a}_n + r_2 \bar{b}_n &= \frac{\rho g a^2}{n} , \\
a_n + b_n + (-c^2)^n \bar{a}_n + (-d^2)^n \bar{b}_n &= 0 ,
\end{aligned}
\tag{68}
$$

and

$$
(-c^2)^n a_n + (-d^2)^n b_n + \bar{a}_n + \bar{b}_n = 0 ,
$$

$$
n = 1, 3, 4, \ldots .
$$

It may be seen similarly that equations (66) are equivalent to the system of equations

$$r_1 a_n + r_2 b_n - r_1(-c^2)^n \bar{a}_n - r_2(-d^2)^n \bar{b}_n = \frac{\rho g a^2}{2n},$$

$$-r_1(-c^2)^n a_n - r_2(-d^2)^n b_n + r_1 \bar{a}_n + r_2 \bar{b}_n = \frac{\rho g a^2}{2n},$$

$$a_n + b_n + (-c^2)^n \bar{a}_n + (-d^2)^n \bar{b}_n = \frac{\rho g a^2}{2n}, \tag{69}$$

and

$$(-c^2)^n a_n + (-d^2)^n b_n + \bar{a}_n + \bar{b}_n = \frac{\rho g a^2}{2n},$$

$$n = 2.$$

The sum of the first two of equations (68) and the sum of the last two of those equations are, respectively,

$$r_1[1 - (-c^2)^n](a_n + \bar{a}_n) + r_2[1 - (-d^2)^n](b_n + \bar{b}_n) = \frac{2\rho g a^2}{n}$$

and (70)

$$[1 + (-c^2)^n](a_n + \bar{a}_n) + [1 + (-d^2)^n](b_n + \bar{b}_n) = 0 ,$$

$$n = 1, 3, 4, \ldots .$$

The respective differences are

$$r_1[1 + (-c^2)^n](a_n - \bar{a}_n) + r_2[1 + (-d^2)^n](b_n - \bar{b}_n) = 0$$

and (71)

$$[1 - (-c^2)^n](a_n - \bar{a}_n) + [1 - (-d^2)^n](b_n - \bar{b}_n) = 0 ,$$

$$n = 1, 3, 4, \ldots .$$

Let

$$a_n = A_n + i\alpha_n \quad \text{and} \quad b_n = B_n + i\beta_n \ , \tag{72}$$

where A_n, α_n, B_n, and β_n are real constants. Then equations (70) and (71) become, after division by 2,

$$r_1[1 - (-c^2)^n]A_n + r_2[1 - (-d^2)^n]B_n = \frac{\rho g a^2}{n} ,$$
$$[1 + (-c^2)^n]A_n + [1 + (-d^2)^n]B_n = 0 , \tag{73}$$

$$r_1[1 + (-c^2)^n]\alpha_n + r_2[1 + (-d^2)^n]\beta_n = 0 ,$$

and

$$[1 - (-c^2)^n]\alpha_n + [1 - (-d^2)^n]\beta_n = 0 , \tag{74}$$

$$n = 1, 3, 4, \ldots .$$

Corresponding equations for n = 2, obtained from equations (69) by the same method as that applied above to equations (68), are

$$r_1[1 - (-c^2)^n]A_n + r_2[1 - (-d^2)^n]B_n = \frac{\rho g a^2}{2n} ,$$
$$[1 + (-c^2)^n]A_n + [1 + (-d^2)^n]B_n = \frac{\rho g a^2}{2n} , \tag{75}$$

$$r_1[1 + (-c^2)^n]\alpha_n + r_2[1 + (-d^2)^n]\beta_n = 0 ,$$

and

$$[1 - (-c^2)^n]\alpha_n + [1 - (-d^2)^n]\beta_n = 0 , \tag{76}$$

$$n = 2.$$

It will be shown that the determinant of the system (73), and hence of the system (75), can equal zero for $n = k$, a natural number, only if A_k and B_k do not affect the stress distribution in the disk and, similarly, that the determinant of the system (74), and hence of the system (76),

can equal zero for $n = k$ only if α_k and β_k do not affect the stresses. Let these determinants be denoted, respectively, by D_n and D'_n, let

$$D_n = \begin{vmatrix} r_1[1 - (-c^2)^n] & r_2[1 - (-d^2)^n] \\ 1 + (-c^2)^n & 1 + (-d^2)^n \end{vmatrix}$$

and (77)

$$D'_n = \begin{vmatrix} r_1[1 + (-c^2)^n] & r_2[1 + (-d^2)^n] \\ 1 - (-c^2)^n & 1 - (-d^2)^n \end{vmatrix} .$$

If $D_n = 0$ for $n = k$, and if we consider $g = 0$, then equations (73) (or equations (75) if $k = 2$) permit arbitrary values of A_k and B_k and hence, if A_k and B_k affect the stresses, allow a stress distribution in the disk when no external forces act. This situation cannot exist physically. Therefore, $D_n = 0$ for $n = k$ implies that A_k and B_k have no part in determining the stresses, or, stated otherwise, if A_k and B_k do affect the stresses, then $D_k \neq 0$. The same argument applied to D'_n shows that $D'_n = 0$ implies that α_n and β_n have no part in determining the stresses, which is to say that if α_n and β_n do affect the stresses, then $D'_n \neq 0$.

To determine which of the constants do affect the stress

distribution and which do not, consider equations (43), which may be rewritten in the form

$$\frac{\partial F}{\partial x} = R\{ \sum_{n=1}^{\infty} (a_n \zeta_1^n + a_{-n}\zeta_1^{-n}) + \sum_{n=1}^{\infty} (b_n \zeta_2^n + b_{-n}\zeta_2^{-n}) + a_o + b_o\}$$

(78)

and

$$\frac{\partial F}{\partial y} = R\{ir_1 \sum_{n=1}^{\infty} (a_n \zeta_1^n + a_{-n}\zeta_1^{-n}) + ir_2 \sum_{n=1}^{\infty} (b_n \zeta_2^n + b_{-n}\zeta_2^{-n}) + ir_1 a_o + ir_2 b_o\} \quad .$$

Since a_o and b_o do not enter into the stresses, given by equations (6), they may be taken equal to zero. With the use of equations (49) and (50), the equations above become (with $a_o = b_o = 0$)

$$\frac{\partial F}{\partial x} = R\{ \sum_{n=1}^{\infty} a_n[\zeta_1^n + (-c^2)^n \zeta_1^{-n}] + \sum_{n=1}^{\infty} b_n[\zeta_2^n + (-d^2)^n \zeta_2^{-n}]\}$$

and (79)

$$\frac{\partial F}{\partial y} = R\{ir_1 \sum_{n=1}^{\infty} a_n[\zeta_1^n + (-c^2)^n \zeta_1^{-n}] + ir_2 \sum_{n=1}^{\infty} b_n[\zeta_2^n + (-d^2)^n \zeta_2^{-n}]\}.$$

Consider now the factor $\zeta_1^n + (-c^2)^n \zeta_1^{-n}$. It will be proved that

$$\zeta_1^n + (-c^2)^n \zeta_1^{-n} = P_n(z_1) \text{ , } n = 1, 2, \ldots , \tag{80}$$

where $P_n(z_1)$ is a polynomial in z_1 of degree n in which the coefficient of z^k, $k = 0, 1, \ldots , n$, depends on both n and k. From equation (30), the first of equations (33), and equation (41), we have

$$z_1 = \frac{a(1 + r_1)}{2}\zeta_1 + \frac{a(1 - r_1)}{2}\zeta_1^{-1}$$

$$= \frac{a(1 + r_1)}{2}\left(\zeta_1 + \frac{1 - r_1}{1 + r_1}\zeta_1^{-1}\right)$$

$$= \frac{a(1 + r_1)}{2}[\zeta_1 + (-c^2)\,\zeta_1^{-1}] ,$$

or

$$\frac{2}{a(1 + r_1)}\,z_1 = \zeta_1 + (-c^2)\,\zeta_1^{-1} \quad . \tag{81}$$

A similar equation can be seen to hold with subscripts 1 replaced by 2 and c replaced by d; thus

$$\frac{2}{a(1 + r_2)}\,z_2 = \zeta_2 + (-d^2)\,\zeta_2^{-1} \quad . \tag{82}$$

Squaring both members of equation (81) and letting

$$k_o = \frac{2}{a(1 + r_1)} \quad , \tag{83}$$

we obtain

$$k_o^2\,z_1^2 = \zeta_1^2 + 2(-c^2) + (-c^2)^2\,\zeta_1^{-2} \; ,$$

or

$$-\,2k_1 + k_o^2\,z_1^2 = \zeta_1^2 + (-c^2)^2\,\zeta_1^{-2} \quad , \tag{84}$$

where

$$k_1 = (-c^2) \; . \tag{85}$$

Thus, equation (80) holds for n = 1, 2. Assume that it holds for n = m, a natural number. To demonstrate that it holds

for $n = m + 1$, we multiply the righthand and lefthand members of equation (80), with $n = m$, by the lefthand and righthand members, respectively, of equation (81) to obtain

$$[P_m(z_1)] \cdot k_o z_1 = [\varsigma_1^m + (-c^2)^m \varsigma_1^{-m}][\varsigma_1 + (-c^2)\,\varsigma_1^{-1}]$$

$$= \varsigma_1^{m+1} + (-c^2)^{m+1} \varsigma_1^{-(m+1)}$$

$$+ (-c^2)[\varsigma_1^{m-1} + (-c^2)^{m-1}\,\varsigma_1^{-(m-1)}]$$

$$= \varsigma_1^{m+1} + (-c^2)^{m+1}\,\varsigma_1^{-(m+1)} + k_1 P_{m-1}(z_1) \quad ,$$

or

$$P_{m+1}(z_1) = \varsigma_1^{m+1} + (-c^2)^{m+1}\,\varsigma_1^{-m+1} \quad , \tag{86}$$

where

$$P_{m+1}(z_1) = k_o z_1 P_m(z_1) - k_1 P_{m-1}(z_1) \quad . \tag{87}$$

It follows, by the principle of mathematical induction, that equation (80) holds for the indicated range of values for n. A similar result obtains for the factors $\varsigma_2^n + (-d^2)^n\,\varsigma_2^{-n}$ and polynomials $P_n'(z_2)$. Therefore, each of the terms in equations (79) for $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ is equal to a polynomial in z_1 or z_2 of degree equal to the index of the term. Since, by equations (6), the stresses are obtained by differentiation of $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, and since $\frac{\partial z_j}{\partial x} = 1$ and $\frac{\partial z_j}{\partial y} = ir_j$, $j = 1, 2$, it follows that all terms with indices greater than 1 will have

a part in determining the stress distribution. That is, all terms containing one of the constants $A_n, \alpha_n, B_n, \beta_n$, with $n = 2, 3, \ldots$, do affect the stress distribution. The contributions to $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ of the terms with index 1 are seen, from equations (79), (81), and (82), to be, respectively,

$$R\{(A_1 + i\alpha_1)\frac{2}{a(1 + r_1)} z_1 + (B_1 + i\beta_1)\frac{2}{a(1 + r_2)} z_2\}$$

and (88)

$$R\{ir_1(A_1 + i\alpha_1)\frac{2}{a(1 + r_1)} z_1 + ir_2(B_1 + i\beta_1)\frac{2}{a(1 + r_2)} z_2\}.$$

Hence, for the terms with index 1, the contribution to $\frac{\partial^2 F}{\partial x^2}$ is

$$\frac{2A_1}{a(1 + r_1)} + \frac{2B_1}{a(1 + r_2)} \quad , \tag{89}$$

the contribution to $\frac{\partial^2 F}{\partial y^2}$ is

$$-\frac{2r_1^2 A_1}{a(1 + r_1)} - \frac{2r_2^2 B_1}{a(1 + r_2)} \quad , \tag{90}$$

and the contribution to $\frac{\partial^2 F}{\partial x \partial y}$ is

$$-\frac{2r_1\alpha_1}{a(1 + r_1)} - \frac{2r_2\beta_1}{a(1 + r_2)} \quad . \tag{91}$$

Referring to equations (6), we see that A_1 and B_1 do affect the stresses σ_x and σ_y, whereas α_1 and β_1 do not, and that the contribution of the terms containing α_1 and β_1 to the stress τ_{xy} is

$$\frac{2}{a}\,\frac{r_1\alpha_1}{1 + r_1} + \frac{r_2\beta_1}{1 + r_2} \quad . \tag{92}$$

It will be shown later (after solution of equations (74)) that this expression is equal to zero, so that α_1 and β_1 do not affect the stresses.

Since A_n and B_n affect the stresses for $n = 1, 2, \ldots$, we conclude from the argument on page 28 that

$$D_n \neq 0, \; n = 1, 2, \ldots , \tag{93}$$

and, since α_n and β_n affect the stresses for $n = 2, 3, \ldots$, we conclude that

$$D_n' \neq 0, \; n = 2, 3, \ldots . \tag{94}$$

Calculation shows that $D_1' = 0$. This in itself implies, by the argument on page 28, that α_1 and β_1 do not affect the stresses, but, as stated below expression (92), this conclusion will be verified later, as a matter of interest.

Simultaneous solution of equations (73) yields

$$A_n = \frac{\rho g a^2}{n}\,\frac{1 + (-d^2)^n}{D_n} \tag{95}$$

and

$$B_n = -\frac{\rho g a^2}{n}\,\frac{1 + (-c^2)^n}{D_n} \; , \tag{96}$$

$$n = 1, 3, 4, \ldots ,$$

where D_n is the determinant given by the first of equations (77) which, upon expansion, may be written

$$D_n = (r_1 - r_2)[1 - (c^2d^2)^n] - (r_1 + r_2)[(-c^2)^n - (-d^2)^n],$$
$$n = 1, 2, \ldots . \qquad (97)$$

The values of A_2 and B_2 obtained by simultaneous solution of equations (75) are

$$A_2 = \frac{\rho g a^2}{4} \frac{1 + d^4 - r_2(1 - d^4)}{D_2}$$

$$= \frac{\rho g a^2}{2} \frac{1 + d^4}{D_2} - \frac{\rho g a^2}{4} \frac{1 + d^4 + r_2(1 - d^4)}{D_2} ,$$

i.e.,

$$A_2 = [A_n, \text{ equation } (95)]_{n=2} - \frac{\rho g a^2}{4} \frac{r_2 + 1 - (r_2 - 1)d^4}{D_2} \qquad (98)$$

and

$$B_2 = \frac{\rho g a^2}{4} \frac{r_1(1 - c^4) - (1 + c^4)}{D_2}$$

$$= -\frac{\rho g a^2}{2} \frac{1 + c^4}{D_2} + \frac{\rho g a^2}{4} \frac{1 + c^4 + r_1(1 - c^4)}{D_2} ,$$

i.e.,

$$B_2' = [B_n, \text{ equation } (96)]_{n=2} + \frac{\rho g a^2}{4} \frac{r_1 + 1 - (r_1 - 1)c^4}{D_2} . \qquad (99)$$

It follows from equations (74) and (76) and statement (94) that

$$\alpha_n = \beta_n = 0, \; n = 2, 3, \ldots . \qquad (100)$$

As stated earlier, $D_1' = 0$, which implies that equations (74)

are dependent for $n = 1$. Solution of either of these equations for α_1, utilizing the definitions (41) and (42) of c and d, yields

$$\alpha_1 = -\frac{r_2(r_1 + 1)}{r_1(r_2 + 1)}\beta_1 \tag{101}$$

where the constant β_1 is arbitrary. Use of equation (101) in expression (92) shows that that expression vanishes identically, which completes the demonstration that α_1 and β_1 do not affect the stresses. It is convenient to set

$$\alpha_1 = \beta_1 = 0 . \tag{102}$$

As a convenient summary of the results of the two preceding paragraphs, the formulas for the values of a_n and b_n, $n = 1, 2, \ldots$, obtained from equations (72), (95) through (100), and (102), are given below.

$$a_n = \frac{\rho g a^2}{n}\,\frac{1 + (-d^2)^n}{D_n}\ , \quad n = 1, 3, 4, \ldots , \tag{103}$$

$$b_n = -\frac{\rho g a^2}{n}\,\frac{1 + (-c^2)^n}{D_n}, \quad n = 1, 3, 4, \ldots , \tag{104}$$

$$a_2 = [a_n, \text{ equation } (103)]_{n=2} - \frac{\rho g a^2}{4}\,\frac{r_2 + 1 - (r_2 - 1)d^4}{D_2}, \tag{105}$$

and

$$b_2 = [b_n, \text{ equation } (104)]_{n=2} + \frac{\rho g a^2}{4}\,\frac{r_1 + 1 - (r_1 - 1)c^4}{D_2}, \tag{106}$$

where

$$D_n = (r_1 - r_2)[1 - (c^2d^2)^n] - (r_1 - r_2)[(-c^2)^n - (-d^2)^n], \quad n = 1, 2, \ldots . \tag{97}$$

With the constants a_n and b_n evaluated, attention is returned to equations (79) for $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$. The differentiation of those equations required by equations (6) in order to obtain the stresses is facilitated by considering first the derivatives of ζ_i with respect to z_i, i = 1, 2. Differentiating equations (32) with respect to their respective independent variables, making use of equations (33), we obtain

$$\frac{d\zeta_i}{dz_i} = \frac{1}{a(r_i + 1)}\left[1 + \frac{z_i}{\pm\sqrt{z_i^2 + a^2(r_i^2 - 1)}}\right]$$

$$= \frac{1}{a(r_i + 1)} \frac{z_i \pm \sqrt{z_i^2 + a^2(r_i^2 - 1)}}{\pm\sqrt{z_i^2 + a^2(r_i^2 - 1)}}$$

$$= \frac{\zeta_i}{W_i}\ , \ i = 1,\ 2;$$

i.e.,

$$\frac{d\zeta_1}{dz_1} = \frac{\zeta_1}{W_1} \quad \text{and} \quad \frac{d\zeta_2}{dz_2} = \frac{\zeta_2}{W_2} \tag{107}$$

Then, since $\frac{\partial z_1}{\partial x} = 1$,

$$\frac{\partial}{\partial x}[\zeta_1^n + (-c^2)^n \zeta_1^{-n}] = [n\zeta_1^{n-1} - n(-c^2)^n \zeta_1^{-n-1}]\frac{\zeta_1}{W_1}$$

$$= \frac{n}{W_1}[\zeta_1^n - (-c^2)^n \zeta_1^{-n}]$$

and, since $\frac{\partial z_1}{\partial y} = ir_1$,

$$\frac{\partial}{\partial y}[\zeta_1^n + (-c^2)^n \zeta_1^{-n}] = [n\zeta_1^{n-1} - n(-c^2)^n \zeta_1^{-n-1}]\frac{\zeta_1}{W_1}\cdot ir_1$$

$$= \frac{ir_1 n}{W_1}[\zeta_1^n - (-c^2)^n \zeta_1^{-n}] \quad . \tag{108}$$

Similarly, it is found that

$$\frac{\partial}{\partial x}[\zeta_2^n + (-d^2)^n \zeta_2^{-n}] = \frac{n}{W_2}[\zeta_2^n - (-d^2)^n \zeta_2^{-n}]$$

and

$$\frac{\partial}{\partial y}[\zeta_2^n + (-d^2)^n \zeta_2^{-n}] = \frac{ir_2 n}{W_2}[\zeta_2^n - (-d^2)^n \zeta_2^{-n}] \; . \tag{109}$$

Now, by the first of equations (6),

$$\sigma_x = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) - \rho g x \quad ,$$

whence, by equations (79), (108), and (109),

$$\sigma_x = -R\left\{\sum_{n=1}^{\infty} n\left[\frac{r_1^2 a_n(\zeta_1^n - (-c^2)^n \zeta_1^{-n})}{W_1} + \frac{r_2^2 b_n(\zeta_2^n - (-d^2)^n \zeta_2^{-n})}{W_2}\right]\right\}$$

$$- \rho g x \; . \tag{110}$$

Similarly, by means of the second and third of equations (6) and by equations (79), (108), and (109), we obtain

$$\sigma_y = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) - \rho g x$$

$$= R\left\{\sum_{n=1}^{\infty} n\left[\frac{a_n(\zeta_1^n - (-c^2)^n \zeta_1^{-n})}{W_1} + \frac{b_n(\zeta_2^n - (-d^2)^n \zeta_2^{-n})}{W_2}\right]\right\}$$

$$- \rho g x \qquad (111)$$

and

$$\tau_{xy} = -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

$$= -R\left\{i\sum_{n=1}^{\infty} n\left[\frac{r_1 a_n(\zeta_1^n - (-c^2)^n \zeta_1^{-n})}{W_1} + \frac{r_2 b_n(\zeta_2^n - (-d^2)^n \zeta_2^{-n}}{W_2}\right]\right\}. \qquad (112)$$

Substitution of the values of a_n and b_n given by equations (103) through (106) into equations (110) through (112) leads to the following formulas for the stresses:

$$\sigma_x = -\rho g a^2 R\left\{\sum_{n=1}^{\infty}\left[\frac{r_1^2(1 + (-d^2)^n)}{W_1 D_n}(\zeta_1^n - (-c^2)^n \zeta_1^{-n})\right.\right.$$

$$\left. - \frac{r_2^2(1 + (-c^2)^n)}{W_2 D_n}(\zeta_2^n - (-d^2)^n \zeta_2^{-n})\right]$$

$$- \frac{r_1^2[r_2 + 1 - (r_2 - 1)d^4]}{2W_1 D_2}(\zeta_1^2 - c^4 \zeta_1^{-2})$$

$$\left. + \frac{r_2^2[r_1 + 1 - (r_1 - 1)c^4]}{2W_2 D_2}(\zeta_2^2 - d^4 \zeta_2^{-2})\right\} - \rho g x, \qquad (113)$$

$$\sigma_y = \rho g a^2 R \left\{ \sum_{n=1}^{\infty} \left[\frac{1 + (-d^2)^n}{W_1 D_n} (\varsigma_1^n - (-c^2)^n \varsigma_1^{-n}) \right.\right.$$

$$\left. - \frac{1 + (-c^2)^n}{W_2 D_n} (\varsigma_2^n - (-d^2)^n \varsigma_2^{-n}) \right]$$

$$- \frac{r_2 + 1 - (r_2 - 1)d^4}{2W_1 D_2} (\varsigma_1^2 - c^4 \varsigma_1^{-2})$$

$$\left. + \frac{r_1 + 1 - (r_1 - 1)c^4}{2W_2 D_2} (\varsigma_2^2 - d^4 \varsigma_2^{-2}) \right\} - \rho g x, \quad (114)$$

and

$$\tau_{xy} = - \rho g a^2 R \left\{ i \sum_{n=1}^{\infty} \left[\frac{r_1(1 + (-d^2)^n)}{W_1 D_n} (\varsigma_1^n - (-c^2)^n \varsigma_1^{-n}) \right.\right.$$

$$\left. - \frac{r_2(1 + (-c^2)^n)}{W_2 D_n} (\varsigma_2^n - (-d^2)^n \varsigma_2^{-n}) \right]$$

$$- \frac{i r_1 [r_2 + 1 - (r_2 - 1)d^4]}{2W_1 D_2} (\varsigma_1^2 - c^4 \varsigma_1^{-2})$$

$$\left. + \frac{i r_2 [r_1 + 1 - (r_1 - 1)c^4]}{2W_2 D_2} (\varsigma_2^2 - d^4 \varsigma_2^{-2}) \right\} ,$$

$$(115)$$

where

$$D_n = (r_1 - r_2)[1 - (c^2 d^2)^n] - (r_1 + r_2)[(-c^2)^n - (-d^2)^n],$$

$$n = 1, 2, \ldots . \quad (97)$$

The similarity of these three formulas allows the writing of the composite equation

$$S_k = \rho g a^2 R\Big\{T_k \sum_{n=1}^{\infty} \frac{1 + (-d^2)^n}{W_1 D_n}(\varsigma_1^n - (-c^2)^n \varsigma_1^{-n})$$

$$- U_k \sum_{n=1}^{\infty} \frac{1 + (-c^2)^n}{W_2 D_n}(\varsigma_2^n - (-d^2)^n \varsigma_2^{-n})$$

$$- T_k \frac{r_2 + 1 - (r_2 - 1)d^4}{2W_1 D_2}(\varsigma_1^2 - c^4 \varsigma_1^{-2})$$

$$+ U_k \frac{r_1 + 1 - (r_1 - 1)c^4}{2W_2 D_2}(\varsigma_2^2 - d^4 \varsigma_2^{-2})\Big\}$$

$$- V_k \cdot \rho g x \,, \quad k = 1, 2, 3, \tag{116}$$

where S_k, T_k, U_k, V_k are given by the table below.

k	S_k	T_k	U_k	V_k
1	σ_x	$-r_1^2$	$-r_2^2$	1
2	σ_y	1	1	1
3	τ_{xy}	ir_1	ir_2	0

This form is more convenient for computation than are the forms of equations (113) through (115).

2.5. Displacements.

The displacements u and v in the disk are obtained from the first two of equations (2),

$$\frac{\partial u}{\partial x} = \epsilon_x = \frac{1}{E_x}\sigma_x - \frac{\nu_{yx}}{E_y}\sigma_y$$

and

$$\frac{\partial v}{\partial y} = \epsilon_y = -\frac{\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y \quad ,$$

by integrating the first with respect to x and the second with respect to y. In view of relation (9) among the constants, these equations may be written

$$\frac{\partial u}{\partial x} = \epsilon_x = \frac{1}{E_x}(\sigma_x - \nu_{xy}\sigma_y)$$

and (117)

$$\frac{\partial v}{\partial y} = \epsilon_y = \frac{1}{E_y}(- \nu_{yx}\sigma_x + \sigma_y)$$

Substitution of the righthand members of equations (110) and (111) for σ_x and σ_y, respectively, leads to

$$\frac{\partial u}{\partial x} = - R\left\{\sum_{n=1}^{\infty} n\left[\left(\frac{r_1^2 + \nu_{xy}}{E_x}\right) \frac{a_n}{W_1}(\zeta_1^n - (-c^2)^n \zeta_1^{-n}) + \left(\frac{r_2^2 + \nu_{xy}}{E_x}\right) \frac{b_n}{W_2}(\zeta_2^n - (-d^2)^n \zeta_2^{-n})\right]\right\} - \left(\frac{1 - \nu_{xy}}{E_x}\right) \rho g x \quad (118)$$

and

$$\frac{\partial v}{\partial y} = R\left\{\sum_{n=1}^{\infty} n\left[\left(\frac{r_1^2\nu_{yx} + 1}{E_y}\right) \frac{a_n}{W_1}(\zeta_1^n - (-c^2)^n \zeta_1^{-n}) + \left(\frac{r_2^2 \nu_{yx} + 1}{E_y}\right) \cdot \frac{b_n}{W_2}(\zeta_2^n - (-d^2)^n \zeta_2^{-n})\right]\right\} + \left(\frac{\nu_{yx} - 1}{E_y}\right)\rho g x. \quad (119)$$

The required integration is facilitated by observing the implications of equations (108) and (109) that

$$\int \frac{\zeta_1^n - (-c^2)^n \zeta_1^{-n}}{W_1}\, dx = \frac{1}{n}[\zeta_1^n + (-c^2)^n \zeta_1^{-n})] + \text{constant},$$

(120)

$$\int \frac{\zeta_1^n - (-c^2)^n \zeta_1^{-n}}{W_1}\, dy = - \frac{1}{r_1 n}[\zeta_1^n + (-c^2)^n \zeta_1^{-n}] + \text{constant},$$

and that similar equations hold for subscripts 2, with c replaced by d. With the use of these relations, equations (118) and (119) are readily integrated to yield

$$u = -R\left\{\sum_{n=1}^{\infty}\left[\frac{r_1^2+\nu_{xy}}{E_x}a_n(\zeta_1^n+(-c^2)^n\zeta_1^{-n})\right.\right.$$

$$\left.\left.+\frac{r_2^2+\nu_{xy}}{E_x}b_n(\zeta_2^n+(-d^2)^n\zeta_2^{-n})\right]\right\}$$

$$-\frac{1-\nu_{xy}}{E_x}\frac{\rho g x^2}{2}+f(y) \tag{121}$$

and

$$v = -R\left\{i\sum_{n=1}^{\infty}\left[\frac{r_1^2\nu_{yx}+1}{r_1E_y}a_n(\zeta_1^n+(-c^2)^n\zeta_1^{-n})\right.\right.$$

$$\left.\left.+\frac{r_2^2\nu_{yx}+1}{r_2E_y}b_n(\zeta_2^n+(-d^2)^n\zeta_2^{-n})\right]\right\}$$

$$+\frac{\nu_{yx}-1}{E_y}\rho g x y+g(x) \quad , \tag{122}$$

where $f(y)$ and $g(x)$ are as yet undetermined functions of their arguments.

It will be shown that comparison of the expression for τ_{xy} which may be obtained from the last one of equations (2) and equations (121) and (122) with that given by equation (112) leads to the determination of $f(y)$ and $g(x)$. Differentiation of equations (121) and (122) with respect to y and x, respectively, with the aid of equations (108) and (109), leads to

$$\frac{\partial u}{\partial y} = - R\left\{i \sum_{n=1}^{\infty} n \left[\frac{r_1^2 + \nu_{xy}}{E_x} \frac{r_1 a_n}{W_1}(\zeta_1^n - (-c^2)^n \zeta_1^{-n})\right.\right.$$

$$\left.\left. + \frac{r_2^2 + \nu_{xy}}{E_x} \frac{r_2 b_n}{W_2}(\zeta_2^n - (-d^2)^n \zeta_2^{-n})\right]\right\} + \frac{df(y)}{dy} \tag{123}$$

and

$$\frac{\partial v}{\partial x} = - R\left\{i \sum_{n=1}^{\infty} n \left[\frac{r_1^2 \nu_{yx} + 1}{r_1 E_y} \frac{a_n}{W_1}(\zeta_1^n - (-c^2)^n \zeta_1^{-n})\right.\right.$$

$$\left.\left. + \frac{r_2^2 \nu_{yx} + 1}{r_2 E_y} \frac{b_n}{W_2}(\zeta_2^n - (-d^2)^n \zeta_2^{-n})\right]\right\}$$

$$+ \frac{\nu_{yx} - 1}{E_y} \rho g y + \frac{dg(x)}{dx} \quad . \tag{124}$$

With the use of these expressions for the derivatives, the last one of equations (2) yields

$$\tau_{xy} = - G_{xy} \cdot R\left\{i \sum_{n=1}^{\infty} n \left[\left(\frac{r_1^2 + \nu_{xy}}{E_x} + \frac{r_1^2 \nu_{yx} + 1}{r_1^2 E_y}\right)\right.\right.$$

$$\frac{r_1 a_n}{W_1}(\zeta_1^n - (-c^2)^n \zeta_1^{-n})$$

$$\left.\left. + \left(\frac{r_2^2 + \nu_{xy}}{E_x} + \frac{r_2^2 \nu_{yx} + 1}{r_2^2 E_y}\right) \frac{r_2 b_n}{W_2}(\zeta_2^n - (-d^2)^n \zeta_1^{-n})\right]\right\}$$

$$+ G_{xy} \frac{\nu_{yx} - 1}{E_y} \rho g y + G_{xy}\left[\frac{df(y)}{dy} + \frac{dg(x)}{dx}\right] \quad . \tag{125}$$

Now, from equation (14),

$$\frac{1}{G_{xy}} = \frac{2K}{\sqrt{E_x E_y}} + \frac{2\nu_{xy}}{E_x} \quad , \tag{126}$$

where, by equations (17), (20), and (33),

$$2K = \alpha^2 + \beta^2 = \alpha^2 + \frac{1}{\alpha^2} = \frac{r_1^2}{\epsilon^2} + \frac{\epsilon^2}{r_1^2} \quad ,$$

or, alternatively, (127)

$$2K = \alpha^2 + \beta^2 = \frac{1}{\beta^2} + \beta^2 = \frac{\epsilon^2}{r_2^2} + \frac{r_2^2}{\epsilon^2} \quad .$$

With $\epsilon^2 = \sqrt{\frac{E_x}{E_y}}$ by equation (12), these last two equations become

$$2K = r_1^2\sqrt{\frac{E_y}{E_x}} + \frac{1}{r_1^2}\sqrt{\frac{E_x}{E_y}}$$

and (128)

$$2K = r_2^2\sqrt{\frac{E_y}{E_x}} + \frac{1}{r_2^2}\sqrt{\frac{E_x}{E_y}} \quad .$$

Insertion of these expressions, in turn, for 2K in equation (126) gives

$$\frac{1}{G_{xy}} = \frac{r_1^2}{E_x} + \frac{1}{r_1^2 E_y} + \frac{2\nu_{xy}}{E_x}$$

and (129)

$$\frac{1}{G_{xy}} = \frac{r_2^2}{E_x} + \frac{1}{r_2^2 E_y} + \frac{2\nu_{xy}}{E_x} \quad .$$

If now, on the authority of equation (9), $\frac{2\nu_{xy}}{E_x}$ be written $\frac{\nu_{xy}}{E_x} + \frac{\nu_{yx}}{E_y}$, then these equations become

$$\frac{1}{G_{xy}} = \frac{r_1^2 + \nu_{xy}}{E_x} + \frac{r_1^2 \nu_{yx} + 1}{r_1^2 E_y}$$

and (130)

$$\frac{1}{G_{xy}} = \frac{r_2^2 + \nu_{xy}}{E_x} + \frac{r_2^2 \nu_{yx} + 1}{r_2^2 E_y} \quad .$$

These equations allow the rewriting of equation (125) as

$$\tau_{xy} = - R\Big\{ i \sum_{n=1}^{\infty} n\Big[\frac{r_1 a_n}{W_1}(\zeta_1^n - (-c^2)^n \zeta_1^{-n})$$

$$+ \frac{r_2 b_n}{W_2}(\zeta_2^n - (-d^2)^n \zeta_2^{-n})\Big]\Big\}$$

$$+ G_{xy}\Big[\frac{\nu_{yx} - 1}{E_y} \rho g y + \frac{df(y)}{dy} + \frac{dg(x)}{dx}\Big] \; . \tag{131}$$

Comparison of this equation with equation (112) indicates that

$$G_{xy}\Big[\frac{\nu_{yx} - 1}{E_y} \rho g y + \frac{df(y)}{dy} + \frac{dg(x)}{dx}\Big] = 0 \; , \tag{132}$$

i.e., since $G_{xy} \neq 0$,

$$\frac{\nu_{yx} - 1}{E_y} \rho g y + \frac{df(y)}{dy} = - \frac{dg(x)}{dx} \quad . \tag{133}$$

Since the lefthand member here is a function of y only, while the righthand member is a function of x only, the fact that the equation must hold for all points (x,y) within the disk leads to the conclusion that both members must be equal to some constant, call it k; then

$$\frac{df(y)}{dy} = - \frac{\nu_{yx} - 1}{E_y} \rho g y + k \ , \tag{134}$$

whence

$$f(y) = - \frac{\nu_{yx} - 1}{E_y} \frac{\rho g y^2}{2} + ky + m \ , \tag{135}$$

where m is an arbitrary constant, and

$$\frac{dg(x)}{dx} = - k \ , \tag{136}$$

whence

$$g(x) = - kx + n \ , \tag{137}$$

where n is an arbitrary constant. Now, the constants m and n represent constant contributions to u and v, as is seen from equations (121) and (122), and hence correspond to a rigid displacement of the whole disk; it follows that, if the support is to be stationary, then m = n = 0. Also, the fact that the displacement u (in the x-direction) is, physically, symmetric with respect to the x-axis requires mathematically, considering equation (121), that f(y) be an even function; it follows, considering equation (135), that k = 0. With k = m = n = 0,

$$f(y) = - \frac{\nu_{yx} - 1}{E_y} \frac{\rho g y^2}{2}$$

and (138)

$$g(x) \equiv 0 \ ,$$

whence equations (121) and (122) become

$$u = -R\{\sum_{n=1}^{\infty}[\frac{r_1^2 + \nu_{xy}}{E_x} a_n(\zeta_1^n + (-c^2)^n \zeta_1^{-n})$$

$$+ \frac{r_2^2 + \nu_{xy}}{E_x} b_n(\zeta_2^n + (-d^2)^n \zeta_2^{-n})]\}$$

$$- \frac{\rho g}{2}(\frac{1 - \nu_{xy}}{E_x} x^2 + \frac{\nu_{yx} + 1}{E_y} y^2) \tag{139}$$

and

$$v = -R\{i \sum_{n=1}^{\infty}[\frac{r_1^2\nu_{yx} + 1}{r_1 E_y} a_n(\zeta_1^n + (-c^2)^n \zeta_1^{-n})$$

$$+ \frac{r_2^2\nu_{yx} + 1}{r_2 E_y} b_n(\zeta_2^n + (-d^2)^n \zeta_2^{-n})]\}$$

$$+ \frac{\nu_{yx} - 1}{E_y} \rho g x y \quad . \tag{140}$$

Inserting the expressions given by equations (103) through (106) for a_n and b_n, we obtain

$$u = -\rho g a^2 R\{\sum_{n=1}^{\infty}[\frac{r_1^2 + \nu_{xy}}{E_x} \frac{(1 + (-d^2)^n)(\zeta_1^n + (-c^2)^n \zeta_1^{-n})}{nD_n}$$

$$- \frac{r_2^2 + \nu_{xy}}{E_x} \frac{(1 + (-c^2)^n)(\zeta_2^n + (-d^2)^n \zeta_2^{-n})}{nD_n}]$$

$$- \frac{r_1^2 + \nu_{xy}}{E_x} \frac{[r_2 + 1 - (r_2 - 1)d^4][\zeta_1^2 + c^4\zeta_1^{-2}]}{4D_2}$$

$$+ \frac{r_2^2 + \nu_{xy}}{E_x} \frac{[r_1 + 1 - (r_1 - 1)c^4][\zeta_2^2 + d^4\zeta_2^{-2}]}{4D_2}\}$$

$$- \frac{\rho g}{2}(\frac{1 - \nu_{xy}}{E_x} x^2 + \frac{\nu_{yx} + 1}{E_y} y^2) \tag{141}$$

and

$$v = -\rho g a^2 R\{i \sum_{n=1}^{\infty}[\frac{r_1^2\nu_{yx}+1}{r_1E_y}\frac{(1+(-d^2)^n)(\zeta_1^n+(-c^2)^n\zeta_1^{-n})}{nD_n}$$

$$-\frac{r_2^2\nu_{yx}+1}{r_2E_y}\frac{(1+(-c^2)^n)(\zeta_2^n+(-d^2)^n\zeta_2^{-n})}{nD_n}]$$

$$-i\frac{r_1^2\nu_{yx}+1}{r_1E_y}\frac{[r_2+1-(r_2-1)d^4][\zeta_1^2+c^4\zeta_1^{-2}]}{4D_2}$$

$$+i\frac{r_2^2\nu_{yx}+1}{r_2E_y}\frac{[r_1+1-(r_1-1)c^4][\zeta_2^2+d^4\zeta_2^{-2}]}{4D_2}\}$$

$$+\frac{\nu_{yx}-1}{E_y}\rho gxy \quad . \qquad (142)$$

2.6. The Isotropic Stress Function as the Limiting Case of the Orthotropic Stress Function.

It is of interest to obtain from the stress function for this problem the stress function for the corresponding problem for isotropic materials. The elastic properties of isotropic materials are independent of direction; thus, $E_x = E_y = E$, $G_{xy} = G$ and $\nu_{xy} = \nu_{yx} = \nu$. By equation (12), then, $\epsilon = 1$. Further, for such materials, the elastic constants are related by the equation[9]

$$\frac{E}{2(1+\nu)} = G \quad . \qquad (143)$$

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[9] Ibid., p. 9.

It follows from this, by equation (14), that

$$K = \frac{E}{2G} - \nu = 1 \quad , \tag{144}$$

whereupon, by equation (17), $\varphi = 0$. With $\epsilon = 1$ and $\varphi = 0$, equations (11), (20), (24), (33), (34), (35), (41), (42), and (97) show that

$$\begin{aligned} z_1 &= z_2 = x + iy = z, \\ r_1 &= r_2 = 1, \\ c &= d = 0, \\ W_1 &= W_2 = z, \\ \zeta_1 &= \zeta_2 = \frac{z}{a}, \end{aligned} \tag{145}$$

and

$$D_n = 0, \; n = 1, 2, \; . \; . \; . \; ,$$

where, in the fourth and fifth equations the positive sign is taken with W_1 and W_2 in order that ζ_1 and ζ_2 be functions of z, instead of vanishing identically.

Insertion of the expressions for a_n and b_n given by equations (103) through (106) into the first of equations (79) leads to

$$\begin{aligned} \frac{\partial F}{\partial x} &= \rho g a^2 R \Big\{ \sum_{n=1}^{\infty} \frac{[1+(-d^2)^n][\zeta_1^n+(-c^2)^n\zeta_1^{-n}] - [1+(-c^2)^n][\zeta_2^n+(-d^2)^n\zeta_2^{-n}]}{nD_n} \\ &- \frac{[r_2+1-(r_2-1)d^4][\zeta_1^2+(-c^2)^2\zeta_1^{-2}] - [r_1+1-(r_1-1)c^4][\zeta_2^2+(-d^2)^2\zeta_2^{-2}]}{4D_2} \Big\}. \end{aligned} \tag{146}$$

For the values of the constants given by equations (145), it

may be seen that the major terms (fractions) in equation (146) are indeterminate. To obtain $\frac{\partial F}{\partial x}$ (and hence, by integration, F) for the isotropic case from equation (146), therefore, we shall set $\epsilon = 1$ in that equation and find, using l'Hospital's rule, the limit of the resulting equation as φ approaches zero. With $\epsilon = 1$,

$$r_1 = e^{\frac{\varphi}{2}} \quad \text{and} \quad r_2 = e^{-\frac{\varphi}{2}}, \tag{147}$$

by equations (20) and (33). The roles of φ in the quantities c, d, z_1, z_2, W_1, W_2, ζ_1, ζ_2, and D_n (with $\epsilon = 1$) are established by the equations defining these quantities, together with relations (147) above. Equations (145) indicate the limits of these quantities as φ approaches zero.

It will prove convenient to take the preliminary step of finding the limits as φ approaches zero of $\frac{\partial \zeta_1}{\partial \varphi}$, $\frac{\partial \zeta_2}{\partial \varphi}$, $\frac{d(-c^2)}{d\varphi}$, and $\frac{d(-d^2)}{d\varphi}$. With $r_1 = e^{\frac{\varphi}{2}}$, with $z_1 = x + ie^{\frac{\varphi}{2}}y$, and with W_1 and ζ_1 given by the first of equations (34) and (35), differentiation yields

$$\frac{\partial \zeta_1}{\partial \varphi} = \frac{\partial \zeta_1}{\partial z_1}\frac{\partial z_1}{\partial \varphi} + \frac{\partial \zeta_1}{\partial r_1}\frac{dr_1}{d\varphi}$$

$$= \frac{\zeta_1}{W_1} \cdot \frac{1}{2} e^{\frac{\varphi}{2}} y + \left[\frac{1}{a(r_1+1)} \cdot \frac{a^2 r_1}{W_1} + (z_1+W_1) \cdot \frac{-1}{a(r_1+1)^2}\right] \frac{1}{2} e^{\frac{\varphi}{2}}, \tag{148}$$

whence, by equations (145), construed as limit equations as

φ approaches zero,

$$\lim_{\varphi \to 0} \frac{\partial \zeta_1}{\partial \varphi} = \frac{iy}{2a} + \left[\frac{a}{2z} - \frac{z}{2a} \right] \frac{1}{2} ,$$

i.e.,

$$\lim_{\varphi \to 0} \frac{\partial \zeta_1}{\partial \varphi} = \frac{a^2 - z^2 + 2iyz}{4az} . \tag{149}$$

Similarly, it may be found that

$$\lim_{\varphi \to 0} \frac{\partial \zeta_2}{\partial \varphi} = - \frac{a^2 - z^2 + 2iyz}{4az} . \tag{150}$$

With $r_1 = e^{\frac{\varphi}{2}}$, $-c^2$ is given by

$$-c^2 = \frac{1 - e^{\frac{\varphi}{2}}}{1 + e^{\frac{\varphi}{2}}} , \tag{151}$$

whence

$$\frac{d(-c^2)}{d\varphi} = \frac{(1 + e^{\frac{\varphi}{2}})(-\frac{1}{2}e^{\frac{\varphi}{2}}) - (1 - e^{\frac{\varphi}{2}})(\frac{1}{2}e^{\frac{\varphi}{2}})}{(1 + e^{\frac{\varphi}{2}})^2} = - \frac{e^{\frac{\varphi}{2}}}{(1 + e^{\frac{\varphi}{2}})^2}, \tag{152}$$

and

$$\lim_{\varphi \to 0} \frac{d(-c^2)}{d\varphi} = - \frac{1}{4} . \tag{153}$$

With $r_2 = e^{-\frac{\varphi}{2}}$, $-d^2$ is given by

$$-d^2 = \frac{1 - e^{-\frac{\varphi}{2}}}{1 + e^{-\frac{\varphi}{2}}} = - \frac{1 - e^{\frac{\varphi}{2}}}{1 + e^{\frac{\varphi}{2}}} = c^2 , \tag{154}$$

whence

$$\lim_{\varphi \to 0} \frac{d(-d^2)}{d\varphi} = \frac{1}{4} . \tag{155}$$

Now,

$$\frac{\partial}{\partial\varphi}[\zeta_1^n + (-c^2)^n\zeta_1^{-n}] = [n\zeta_1^{n-1} - (-c^2)^n n\zeta_1^{-n-1}]\frac{\partial\zeta_1}{\partial\varphi}$$

$$+ n(-c^2)^{n-1}\zeta_1^{-n}\frac{d(-c^2)}{d\varphi} \quad . \qquad (156)$$

Hence, by means of equations (145), construed as limit equations as φ approaches zero, and equations (149) and (153),

$$\lim_{\varphi\to 0}\frac{\partial}{\partial\varphi}[\zeta_1^n + (-c^2)^n\zeta_1^{-n}]$$

$$= n[(\frac{z}{a})^{n-1} - 0]\frac{a^2 - z^2 + 2iyz}{4az} + \frac{a}{z}(-\frac{1}{4})^* \quad ,$$

i.e.,

$$\lim_{\varphi\to 0}\frac{\partial}{\partial\varphi}[\zeta_1^n + (-c^2)^n\zeta_1^{-n}]$$

$$= \frac{n(a^2z^{n-2} - z^n + 2iyz^{n-1})}{4a^n} - \frac{a}{4z}^* \quad , \qquad (157)$$

where the star (*) beside a term is used to indicate that that term is present for n = 1 only, vanishing for all other values of n. Similarly, it can be shown that

$$\lim_{\varphi\to 0}\frac{\partial}{\partial\varphi}[\zeta_2^n + (-d^2)^n\zeta_2^{-n}]$$

$$= -\frac{n(a^2z^{n-2} - z^n + 2iyz^{n-1})}{4a^n} + \frac{a}{4z}^* \quad . \qquad (158)$$

Let the numerator of the nth term of the summation in equation (146) be denoted by N_n, i.e., let

$$N_n = [1 + (-d^2)^n][\zeta_1^n + (-c^2)^n\zeta_1^{-n}]$$

$$- [1 + (-c^2)^n][\zeta_2^n + (-d^2)^n\zeta_2^{-n}] \quad . \qquad (159)$$

Then

$$\frac{\partial N_n}{\partial \varphi} = [1 + (-d^2)^n] \frac{\partial}{\partial \varphi} [\varsigma_1^n + (-c^2)^n \varsigma_1^{-n}]$$

$$+ [\varsigma_1^n + (-c^2)^n \varsigma_1^{-n}][n(-d^2)^{n-1} \frac{d(-d^2)}{d\varphi}]$$

$$- [1 + (-c^2)^n] \frac{\partial}{\partial \varphi} [\varsigma_2^n + (-d^2)^n \varsigma_2^{-n}]$$

$$- [\varsigma_2^n + (-d^2)^n \varsigma_2^{-n}][n(-c^2)^{n-1} \frac{d(-c^2)}{d\varphi}] \quad , \qquad (160)$$

and, by equations (145), (153), (155), (157), and (158),

$$\lim_{\varphi \to 0} \frac{\partial N_n}{\partial \varphi} = \frac{n(a^2 z^{n-2} - z^n + 2iyz^{n-1})}{4a^n} - \frac{a}{4z}^{\star} + \frac{z}{a}\left(\frac{1}{4}\right)^{\star}$$

$$+ \frac{n(a^2 z^{n-2} - z^n + 2iyz^{n-1})}{4a^n} - \frac{a}{4z}^{\star} - \frac{z}{a}\left(-\frac{1}{4}\right)^{\star} \quad ,$$

i.e.,

$$\lim_{\varphi \to 0} \frac{\partial N_n}{\partial \varphi} = \frac{n(a^2 z^{n-2} - z^n + 2iyz^{n-1})}{2a^n} - \frac{a}{2z}^{\star} + \frac{z}{2a}^{\star} \quad . \qquad (161)$$

From equation (97), we obtain, by differentiation with respect to φ,

$$\frac{d(D_n)}{d\varphi} = [r_1 - r_2] \frac{d}{d\varphi}[1 - (c^2 d^2)^n] + [1 - (c^2 d^2)^n][\tfrac{1}{2}e^{\frac{\varphi}{2}} + \tfrac{1}{2}e^{-\frac{\varphi}{2}}]$$

$$- [r_1 + r_2][n(-c^2)^{n-1} \frac{d(-c^2)}{d\varphi} - n(-d^2)^{n-1} \frac{d(-d^2)}{d\varphi}]$$

$$- [(-c^2)^n - (-d^2)^n][\frac{d(r_1 + r_2)}{d\varphi}] \quad , \qquad (162)$$

wherein equations (147), $r_1 = e^{\frac{\varphi}{2}}$ and $r_2 = e^{-\frac{\varphi}{2}}$, have been used to obtain $\frac{d}{d\varphi}(r_1 - r_2)$ for the second term. Hence, by the second

and third of equations (145) and by equations (153) and (155),

$$\lim_{\varphi \to 0} \frac{d(D_n)}{d\varphi} = 1 - 2(-\tfrac{1}{4} - \tfrac{1}{4})^* \quad ,$$

i.e.,

$$\lim_{\varphi \to 0} \frac{d(D_n)}{d\varphi} = \begin{cases} 2, & \text{for } n = 1 \\ 1, & \text{for } n = 2, 3, \ldots . \end{cases} \tag{163}$$

By l'Hospital's rule, now, the limit, as φ approaches zero, of the nth term of the summation in equation (146) is, provided all the limits exist,

$$\lim_{\varphi \to 0} \frac{N_n}{nD_n} = \frac{1}{n} \lim_{\varphi \to 0} \frac{N_n}{D_n} = \frac{1}{n} \lim_{\varphi \to 0} \frac{\frac{\partial N_n}{\partial \varphi}}{\frac{d(D_n)}{d\varphi}} = \frac{\lim\limits_{\varphi \to 0} \frac{\partial N_n}{\partial \varphi}}{n \lim\limits_{\varphi \to 0} \frac{d(D_n)}{d\varphi}} ; \tag{164}$$

hence, by equations (161) and (163),

$$\lim_{\varphi \to 0} \frac{N_n}{nD_n} = \begin{cases} \dfrac{a^2 - z^2 + 2iyz}{4az} - \dfrac{a}{4z} + \dfrac{z}{4a}, & \text{for } n = 1 \\[2ex] \dfrac{a^2 z^{n-2} - z^n + 2iyz^{n-1}}{2a^n}, & \text{for } n = 2, 3, \ldots . \end{cases} \tag{165}$$

Consider now the last term in equation (146), which may be denoted by $-\frac{N_o}{4D_2}$, where

$$N_o = [r_2 + 1 - (r_2 - 1)d^4][\zeta_1^2 + (-c^2)^2 \zeta_1^{-2}] - [r_1 + 1 - (r_1 - 1)c^4][\zeta_2^2 + (-d^2)^2 \zeta_2^{-2}]. \tag{166}$$

Then

$$\frac{\partial N_o}{\partial \varphi} = [r_2 + 1 - (r_2 - 1)d^4]\,\frac{\partial}{\partial \varphi}\,[\zeta_1^2 + (-c^2)^2\zeta_1^{-2}]$$

$$+ [\zeta_1^2 + (-c^2)^2\zeta_1^{-2}][-\tfrac{1}{2}e^{-\frac{\varphi}{2}} - (r_2 - 1)2d^2\,\frac{d(d^2)}{d\varphi}$$

$$- d^4(-\tfrac{1}{2}e^{-\frac{\varphi}{2}})]$$

$$- [r_1 + 1 - (r_1 - 1)c^4]\,\frac{\partial}{\partial \varphi}\,[\zeta_2^2 + (-d^2)^2\zeta_2^{-2}]$$

$$- [\zeta_2^2 + (-d^2)^2\zeta_2^{-2}][\tfrac{1}{2}e^{\frac{\varphi}{2}} - (r_1 - 1)2c^2\,\frac{d(c^2)}{d\varphi}$$

$$- c^4 \cdot \tfrac{1}{2}e^{\frac{\varphi}{2}}] \,, \qquad (167)$$

wherein equations (147), $r_1 = e^{\frac{\varphi}{2}}$ and $r_2 = e^{-\frac{\varphi}{2}}$, have been used as needed for differentiation. It follows, by equations (145), (153), (155), (157) and (158), that

$$\lim_{\varphi \to 0} \frac{\partial N_o}{\partial \varphi} = [2][\frac{a^2 - z^2 + 2ivz}{2a^2}] + [(\tfrac{z}{a})^2][-\tfrac{1}{2}]$$

$$- [2][-\frac{a^2 - z^2 + 2ivz}{2a^2}] - [(\tfrac{z}{a})^2][\tfrac{1}{2}]$$

$$= \frac{2(a^2 - z^2 + 2ivz)}{a^2} - (\tfrac{z}{a})^2 \quad ,$$

i.e.,

$$\lim_{\varphi \to 0} \frac{\partial N_o}{\partial \varphi} = -\frac{3z^2}{a^2} + \frac{4ivz}{a^2} + 2 \quad . \qquad (168)$$

Hence, by l'Hospital's rule

$$\lim_{\varphi\to 0} -\frac{N_0}{4D_2} = -\frac{1}{4}\lim_{\varphi\to 0}\frac{N_0}{D_2} = -\frac{1}{4}\lim_{\varphi\to 0}\frac{\frac{\partial N_0}{\partial\varphi}}{\frac{d(D_2)}{d\varphi}} = -\frac{\lim_{\varphi\to 0}\frac{\partial N_0}{\partial\varphi}}{4\lim_{\varphi\to 0}\frac{d(D_2)}{d\varphi}}$$

$$= -\frac{3}{4}\frac{z^2}{a^2} + \frac{iyz}{a^2} + \frac{1}{2} \quad , \tag{169}$$

equation (163) being used, with equation (168) in the last step.

Now, from equations (146), (159), (165), (166), and (169), it can be stated that

$$\lim_{\varphi\to 0}\frac{\partial F}{\partial x} = \rho g a^2 \ R\{\sum_{n=1}^{\infty}[\lim_{\varphi\to 0}\frac{N_n}{nD_n}] - \lim_{\varphi\to 0}\frac{N_0}{4D_2}\}$$

$$= \rho g a^2 \ R\{\sum_{n=2}^{\infty}[\frac{a^2 z^{n-2} - z^n + 2iyz^{n-1}}{2a^n}] + \frac{a^2 - z^2 + 2iyz}{4az}$$

$$- \frac{a}{4z} + \frac{z}{4a} + \frac{3z^2}{4a^2} - \frac{iyz}{a^2} - \frac{1}{2}\} \quad . \tag{170}$$

This may be written

$$\lim_{\varphi\to 0}\frac{\partial F}{\partial x} = \rho g a^2 \cdot R\{\sum_{n=2}^{\infty}\frac{1}{2a^n}[a^2 z^{n-2} - z^n + 2iyz^{n-1}]$$

$$+ \frac{3}{4}\frac{z^2}{a^2} - \frac{iyz}{a^2} + \frac{iy}{2a} - \frac{1}{2}\} \quad . \tag{171}$$

Integration with respect to x yields

$$\lim_{\varphi\to 0} F = \rho g a^2 \ R\{\sum_{n=2}^{\infty}\frac{1}{2a^n}[a^2\frac{z^{n-1}}{n-1} - \frac{z^{n+1}}{n+1} + 2iy\frac{z^n}{n}]$$

$$+ \frac{z^3}{4a^2} - \frac{iyz^2}{2a^2} + \frac{iyz}{2a} - \frac{z}{2}\} \quad , \tag{172}$$

where the constant of integration, which does not affect the stresses, has been taken to be zero. This result may be simplified as follows:

$$\lim_{\varphi \to 0} F = \rho g a^2 \, R\{\sum_{n=2}^{\infty} \frac{1}{2a^n}[\frac{z^{n+1}}{n+1} - \frac{\bar{z}^{n+1}}{n+1} + 2iy\,\frac{z^n}{n}] + \frac{z}{2} + \frac{z^2}{4a} + \frac{z^3}{4a^2}$$

$$- \frac{iyz^2}{2a^2} + \frac{iyz}{2a} - \frac{z}{2}\}$$

$$= \rho g a^2 \, R\{\sum_{n=1}^{\infty} \frac{1}{2a^n}[\frac{2iyz^{n+1}}{n+1}] + \frac{z^3}{4a^2} + \frac{z^2}{4a} - \frac{(z - \bar{z})z^2}{4a^2}$$

$$+ \frac{(z - \bar{z})z}{4a}\}$$

$$= \rho g a \, R\{\sum_{n=1}^{\infty} \frac{(z - \bar{z})z^{n+1}}{2(n + 1)a^n} + \frac{z^2}{2} + \frac{z^2\bar{z}}{4a} - \frac{z\bar{z}}{4}\} \quad ;$$

i.e.,

$$\lim_{\varphi \to 0} F = \frac{\rho g a}{2} \, R\{\sum_{n=1}^{\infty} \frac{(z - \bar{z})z^{n+1}}{(n + 1)a^n} + z^2 + \frac{z^2\bar{z}}{2a} - \frac{z\bar{z}}{2}\} \quad . \qquad (173)$$

This function, $\lim_{\varphi \to 0} F$, is, in fact, the stress function for an isotropic disk under the same loading as that considered for the orthotropic disk.

CHAPTER III

STRESS DISTRIBUTION IN AN ORTHOTROPIC DISK SUBJECTED TO ITS OWN WEIGHT WHEN SUPPORTED AT ITS CENTER

3.1. Plan for Solution.

Consider an orthotropic disk of radius a, referred to a rectangular coordinate system as described in section 2.1. The stress distribution in the disk subjected to its own weight and supported at its center will be found by (1) finding the stresses due only to the supporting force, $P = \rho g \pi a^2$, with no gravity (weight) force, and (2) superimposing on this stress distribution the stresses due to the weight of the disk, with the requirement that the resultant stresses be zero on the boundary of the disk. Section 3.2 will be concerned with the first case, section 3.3 with the second. Each of these cases, like the problem of Chapter I, consists essentially of finding a solution to equation (13) in the form of equation (23) which satisfies equations (26) on the boundary of the disk. In the first case, of course, the equations referred to above must be modified by setting $g = 0$.

3.2. Stresses in the Disk Due Only to the Supporting Force P at Its Center.

Let the stress function, the stresses, and the displacements corresponding to the single force P at the center be distinguished by primes ('). With $g = 0$, then, equations (6) become

$$\sigma_x' = \frac{\partial^2 F'}{\partial y^2}\,,\quad \sigma_y' = \frac{\partial^2 F'}{\partial x^2}\,,\quad \tau_{xy}' = -\frac{\partial^2 F'}{\partial x \partial y} \tag{174}$$

and equations (26) become

$$X_\nu' = \frac{d}{ds}\left(\frac{\partial F'}{\partial y}\right)$$

and (175)

$$Y_\nu' = -\frac{d}{ds}\left(\frac{\partial F'}{\partial x}\right) \quad .$$

A suitable stress function for this case is given by

$$F' = R\{Az_1 \log z_1 + Bz_2 \log z_2\} \quad , \tag{176}$$

where z_1 and z_2 are defined by equations (24) and A and B are complex coefficients. The constants A and B are determined by the following conditions:

(1) the stress σ_x' is symmetric with respect to the x-axis,

(2) the integral $\int_C X_\nu' ds$, taken around the boundary C of the disk, representing the resultant of the x-components of the boundary stresses, is equal to the magnitude P of the applied load, and

(3) the displacements u' and v', obtained by integration from equations (2), are single-valued.

These conditions will be applied in succession in the paragraphs which follow.

Differentiation of F' gives

$$\frac{\partial F'}{\partial x} = R\{A(1 + \log z_1) + B(1 + \log z_2)\}$$

and (177)

$$\frac{\partial F'}{\partial y} = R\{ir_1A(1 + \log z_1) + ir_2B(1 + \log z_2)\} \quad ,$$

and, again,

$$\frac{\partial^2 F'}{\partial x^2} = R\{\frac{A}{z_1} + \frac{B}{z_2}\} \quad ,$$

$$\frac{\partial^2 F'}{\partial y^2} = - R\{\frac{r_1^2 A}{z_1} + \frac{r_2^2 B}{z_2}\} \quad , \qquad (178)$$

and

$$\frac{\partial^2 F'}{\partial x \partial y} = R\{\frac{ir_1A}{z_1} + \frac{ir_2B}{z_2}\} \quad .$$

Let

$$A = A' + i\alpha' \quad \text{and} \quad B = B' + i\beta' \ , \qquad (179)$$

where A', B', α', and β' are real constants. Then, by the first of equations (174) and the second of equations (178), with $\frac{1}{z_1}$ and $\frac{1}{z_2}$ replaced by their respective equals $\frac{x - ir_1y}{x^2 + r_1^2y^2}$ and $\frac{x - ir_2y}{x^2 + r_2^2y^2}$,

$$\sigma_x' = - R\{\frac{r_1^2(A' + i\alpha')(x - ir_1y)}{x^2 + r_1^2y^2} + \frac{r_2^2(B' + i\beta')(x - ir_2y)}{x^2 + r_2^2y^2}\} \ ,$$

i.e.,

$$\sigma_x' = -r_1^2 \frac{A'x + r_1\alpha' y}{x^2 + r_1^2 y^2} - r_2^2 \frac{B'x + r_2\beta' y}{x^2 + r_2^2 y^2} \quad . \tag{180}$$

Condition (1), that σ_x' be symmetric with respect to the x-axis, is seen now to require that (since we have assumed $r_1 \neq r_2$)

$$\alpha' = \beta' = 0 \quad . \tag{181}$$

Consider next condition (2), $\int_C X_\nu' \, ds = P$, where C is the boundary of the disk. By the first of equations (175), this condition allows us to write

$$\int_C \frac{d}{ds}\left(\frac{\partial F'}{\partial y}\right) ds = \int_{C_1} \frac{\partial}{\partial z_1}\left(\frac{\partial F'}{\partial y}\right) dz_1 + \int_{C_2} \frac{\partial}{\partial z_2}\left(\frac{\partial F'}{\partial y}\right) dz_2 = P \ , \tag{182}$$

whence, by the second of equations (177),

$$R\left\{ir_1 A \int_{C_1} \frac{dz_1}{z_1} + ir_2 B \int_{C_2} \frac{dz_2}{z_2}\right\} = P \ , \tag{183}$$

where C_1 and C_2 are the ellipses in the z_1- and z_2-planes, respectively, corresponding to C in the z-plane (the plane of the disk). The residues of the integrands $\frac{1}{z_1}$ and $\frac{1}{z_2}$ at $z_1 = 0$ and $z_2 = 0$, respectively, are both equal to 1, so that application of Cauchy's residue theorem to the integrals of equation (183) yields

$$R\{ir_1 A \cdot 2\pi i + ir_2 B \cdot 2\pi i\} = P \ , \tag{184}$$

or since $A = A'$ (real) and $B = B'$ (real) by equations (179) and (181),

$$r_1 A + r_2 B = -\frac{P}{2\pi} = -\frac{\rho g a^2}{2} \quad . \tag{185}$$

The last equality here results from the observation that $P = \rho g \pi a^2$.

By the first two of equations (2) and the first two of equations (174),

$$\frac{\partial u'}{\partial x} = \frac{1}{E_x}\frac{\partial^2 F'}{\partial y^2} - \frac{\nu_{yx}}{E_y}\frac{\partial^2 F'}{\partial x^2}$$

and (186)

$$\frac{\partial v'}{\partial y} = -\frac{\nu_{xy}}{E_x}\frac{\partial^2 F'}{\partial y^2} + \frac{1}{E_y}\frac{\partial^2 F'}{\partial x^2} \quad .$$

Substitution of the expressions given by the first two of equations (178) for $\frac{\partial^2 F'}{\partial x^2}$ and $\frac{\partial^2 F'}{\partial y^2}$ in equations (186) leads to

$$\frac{\partial u'}{\partial x} = R\{-A(\frac{r^2}{E_x} + \frac{\nu_{yx}}{E_y})\frac{1}{z_1} - B(\frac{r_2^2}{E_x} + \frac{\nu_{yx}}{E_y})\frac{1}{z_2}\}$$

and (187)

$$\frac{\partial v'}{\partial y} = R\{A(\frac{r_1^2\nu_{xy}}{E_x} + \frac{1}{E_y})\frac{1}{z_1} + B(\frac{r_2^2\nu_{xy}}{E_x} + \frac{1}{E_y})\frac{1}{z_2}\} \quad .$$

Integration of the first of these equations with respect to x and the second with respect to y yields

$$u' = R\{-A(\frac{r_1^2}{E_x} + \frac{\nu_{yx}}{E_y})(\log|z_1| + i\theta')$$
$$- B(\frac{r_2^2}{E_x} + \frac{\nu_{yx}}{E_y})(\log|z_2| + i\varphi')\}$$

and (188)

$$v' = R\{A(\frac{r_1^2\nu_{xy}}{E_x} + \frac{1}{E_y})\frac{1}{ir_1}(\log|z_1| + i\theta')$$
$$+ B(\frac{r_2^2\nu_{xy}}{E_x} + \frac{1}{E_y})\frac{1}{ir_2}(\log|z_2| + i\varphi')\},$$

where θ' and φ' are the (multiple-valued) arguments of z_1 and z_2, respectively. Since A and B are real, by equations (179) and (181), equations (188) may be written

$$u' = -A\left(\frac{r_1^2}{E_x} + \frac{\nu_{yx}}{E_y}\right) \log |z_1| - B\left(\frac{r_2^2}{E_x} + \frac{\nu_{yx}}{E_y}\right) \log |z_2|$$

and (189)

$$v' = A\left(\frac{r_1 \nu_{xy}}{E_x} + \frac{1}{r_1 E_y}\right)\theta' + B\left(\frac{r_2 \nu_{xy}}{E_x} + \frac{1}{r_2 E_y}\right)\varphi' \quad .$$

The displacement u' as given here is clearly single-valued. Since, $\theta' = \text{Arc tan } \frac{r_1 y}{x} + 2n\pi$ and $\varphi' = \text{Arc tan } \frac{r_2 y}{x} + 2n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$, the single-valuedness of v' requires that

$$A\left(\frac{r_1 \nu_{xy}}{E_x} + \frac{1}{r_1 E_y}\right) + B\left(\frac{r_2 \nu_{xy}}{E_x} + \frac{1}{r_2 E_y}\right) = 0 \quad . \tag{190}$$

Equations (185) and (190), solved simultaneously, yield expressions for A and B which may be put into the form

$$A = -\frac{\rho g a^2 r_1 (E_x + r_2^2 \nu_{xy} E_y)}{2E_x(r_1^2 - r_2^2)}$$

and (191)

$$B = \frac{\rho g a^2 r_2 (E_x + r_1^2 \nu_{xy} E_y)}{2E_x(r_1^2 - r_2^2)} \quad .$$

Replacement of $\nu_{xy}E_y$ by $\nu_{yx}E_x$, permissible by equation (9), leads to the simpler forms

$$A = -\frac{\rho g a^2 r_1 (1 + r_2^2 \nu_{yx})}{2(r_1^2 - r_2^2)}$$

and (192)

$$B = \frac{\rho g a^2 r_2 (1 + r_1^2 \nu_{yx})}{2(r_1^2 - r_2^2)} \quad .$$

With A and B determined, the stresses due to the load P may be found from equations (174) and (178). Thus, since A and B are real and since $\frac{1}{z_1} = \frac{x - ir_1 y}{x^2 + r_1^2 y^2}$ and $\frac{1}{z_2} = \frac{x - ir_2 y}{x^2 + r_2^2 y^2}$,

$$\sigma_x' = -\frac{r_1^2 Ax}{x^2 + r_1^2 y^2} - \frac{r_2^2 Bx}{x^2 + r_2^2 y^2} \,,$$

$$\sigma_y' = \frac{Ax}{x^2 + r_1^2 y^2} + \frac{Bx}{x^2 + r_2^2 y^2} \,, \tag{193}$$

and

$$\tau_{xy}' = -\frac{r_1^2 Ay}{x^2 + r_1^2 y^2} - \frac{r_2^2 By}{x^2 + r_2^2 y^2} \,,$$

where A and B are the constants given by equations (192). The boundary stress components X_ν' and Y_ν' may be obtained from equations (175) and (177). There is no need for explicit expressions for these quantities at this point, however, and their derivation will not be carried out here.

3.3. Superposition of the Stresses Due to the Weight of the Disk.

Inasmuch as this phase of the problem differs essentially from the problem of Chapter II only in the boundary conditions, the same stress function as that of Chapter II, except for the values of the constants, will be assumed here; i.e., the stresses due to the weight of the disk may be obtained

from equations (6) and equations (43) once the constants a_n and b_n have been evaluated. In fact, with a_n and b_n evaluated, these stresses may be obtained directly by insertion of the new values for a_n and b_n in equations (110) through (112). The determining condition for the constants here is that the resultant of this stress distribution and that due to the force P at the center of the disk must be such that $X_\nu = Y_\nu = 0$ on the boundary, where, from equations (26) and (175),

$$X_\nu = \frac{d}{ds}\left(\frac{\partial F}{\partial y} + \frac{\partial F'}{\partial y}\right) - \rho g x \frac{dy}{ds}$$

and (194)

$$Y_\nu = -\frac{d}{ds}\left(\frac{\partial F}{\partial x} + \frac{\partial F'}{\partial x}\right) + \rho g x \frac{dx}{ds} \quad .$$

With $X_\nu = Y_\nu = 0$, these equations may be written

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) = -\frac{d}{ds}\left(\frac{\partial F'}{\partial y}\right) + \rho g x \frac{dy}{ds}$$

and (195)

$$\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) = -\frac{d}{ds}\left(\frac{\partial F'}{\partial x}\right) + \rho g x \frac{dx}{ds} \quad ,$$

or, with the aid of equations (54) and (56),

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) = -\frac{d}{ds}\left(\frac{\partial F'}{\partial y}\right) + \frac{\rho g a^2}{2}(1 + \cos 2\theta)\frac{d\theta}{ds}$$

and (196)

$$\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) = -\frac{d}{ds}\left(\frac{\partial F'}{\partial x}\right) - \frac{\rho g a^2}{2}\sin 2\theta \frac{d\theta}{ds} \quad .$$

Integration gives now

$$\frac{\partial F}{\partial y} = -\frac{\partial F'}{\partial y} + \frac{\rho g a^2}{4}(2\theta + \sin 2\theta)$$

and (197)

$$\frac{\partial F}{\partial x} = -\frac{\partial F'}{\partial x} + \frac{\rho g a^2}{4}\cos 2\theta \quad ,$$

where the constants of integration, which do not affect the stresses, have been taken to be zero.

It will prove convenient to consider, from equations (197), the equation

$$\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = [-\frac{\partial F'}{\partial x} + \frac{\rho g a^2}{4}\cos 2\theta] + i[-\frac{\partial F'}{\partial y} + \frac{\rho g a^2}{4}(2\theta + \sin 2\theta)] \quad . \tag{198}$$

If in equations (177) the constant terms be neglected, inasmuch as they do not affect the stresses obtained from F', then those equations become

$$\frac{\partial F'}{\partial x} = R\{A \log z_1 + B \log z_2\}$$

and (199)

$$\frac{\partial F'}{\partial y} = R\{ir_1 A \log z_1 + ir_2 B \log z_2\} \quad .$$

Since A and B are real coefficients, by equations (192), and since, on the boundary

$$\log z_1 = \log a\sqrt{\cos^2\theta + r_1^2 \sin^2\theta} + i \arc\tan\left(\frac{r_1 \sin\theta}{\cos\theta}\right)$$

and (200)

$$\log z_2 = \log a\sqrt{\cos^2\theta + r_2^2 \sin^2\theta} + i \arc\tan\left(\frac{r_2 \sin\theta}{\cos\theta}\right) \quad ,$$

equations (199) may be written, for the boundary of the disk,

$$\frac{\partial F'}{\partial x} = A \log \left(a\sqrt{\cos^2\theta + r_1^2 \sin^2\theta}\right)$$

$$+ B \log \left(a\sqrt{\cos^2\theta + r_2^2 \sin^2\theta}\right)$$

and (201)

$$\frac{\partial F'}{\partial y} = - r_1 \arc\tan \left(\frac{r_1 \sin\theta}{\cos\theta}\right) - r_2 \arc\tan \left(\frac{r_2 \sin\theta}{\cos\theta}\right).$$

Substitution of these expressions for $\frac{\partial F'}{\partial x}$ and $\frac{\partial F'}{\partial y}$ in equations (198) gives

$$\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y} = f(\theta) + ig(\theta)\ , \tag{202}$$

where

$$f(\theta) + ig(\theta) = \Big[-A \log \left(a\sqrt{\cos^2\theta + r_1^2 \sin^2\theta}\right)$$

$$-B \log \left(a\sqrt{\cos^2\theta + r_2^2 \sin^2\theta}\right)$$

$$+ \frac{\rho g a^2}{4} \cos 2\theta\Big] + i\Big[r_1 A \arc\tan \left(\frac{r_1 \sin\theta}{\cos\theta}\right)$$

$$+ r_2 B \arc\tan \left(\frac{r_2 \sin\theta}{\cos\theta}\right) + \frac{\rho g a^2}{4}(2\theta + \sin 2\theta)\Big],$$

(203)

or, since $\cos 2\theta + i \sin 2\theta = e^{2i\theta}$,

$$f(\theta) + ig(\theta) = - A \log \left(a\sqrt{\cos^2\theta + r_1^2 \sin^2\theta}\right)$$

$$- B \log \left(a\sqrt{\cos^2\theta + r_2^2 \sin^2\theta}\right)$$

$$+ ir_1 A \arc\tan \left(\frac{r_1 \sin\theta}{\cos\theta}\right)$$

$$+ ir_2 B \arc\tan \left(\frac{r_2 \sin\theta}{\cos\theta}\right)$$

$$+ \frac{\rho g a^2}{4}\, e^{2i\theta} + i\,\frac{\rho g a^2}{2}\,\theta\ . \tag{204}$$

The function $f(\theta) + ig(\theta)$ can be represented by a complex Fourier series; then equation (202) becomes

$$\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y} = \sum_{-\infty}^{\infty} c_n\, e^{ni\theta} \quad , \tag{205}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(\theta) + ig(\theta)]e^{-ni\theta}\, d\theta \ ,$$
$$n = 0, \pm 1, \pm 2, \ldots . \tag{206}$$

For $n = 0, \pm 1, \pm 2, \ldots$, let

$$I_1 = \int_{-\pi}^{\pi} [\log (a\sqrt{\cos^2\theta + r_1^2 \sin^2\theta}\,)]e^{-ni\theta}\, d\theta \ ,$$

$$I_2 = \int_{-\pi}^{\pi} [\log (a\sqrt{\cos^2\theta + r_2^2 \sin^2\theta}\,)]e^{-ni\theta}\, d\theta \ ,$$

$$I_3 = \int_{-\pi}^{\pi} [\arctan (\frac{r_1 \sin\theta}{\cos\theta})]e^{-ni\theta}\, d\theta \ ,$$

$$I_4 = \int_{-\pi}^{\pi} [\arctan (\frac{r_2 \sin\theta}{\cos\theta})]e^{-ni\theta}\, d\theta \ , \tag{207}$$

$$I_5 = \int_{-\pi}^{\pi} e^{-(n-2)i\theta}\, d\theta \ ,$$

and

$$I_6 = \int_{-\pi}^{\pi} \theta e^{-ni\theta}\, d\theta \ .$$

Then, from equations (204) and (206),

$$c_n = \frac{1}{2\pi}\{- A \cdot I_1 - B \cdot I_2 + ir_1 A \cdot I_3 + ir_2 B \cdot I_4$$
$$+ \frac{\rho g a^2}{4} \cdot I_5 + i\,\frac{\rho g a^2}{2} \cdot I_6\} \ ,$$
$$n = 0, \pm 1, \pm 2, \ldots . \tag{208}$$

The integrals I_1 through I_6 will be evaluated successively in

the following paragraphs.

In the evaluation of I_1 it will be necessary to have the value of $\int_0^{2\pi} \frac{\cos k\varphi}{a-b\cos\varphi}\,d\varphi$, where $k = 0, 1, 2, \ldots$, and a and b are real constants such that $a > 0$ and $a^2 > b^2$. To this end, consider

$$\int_C \frac{z^k}{z^2 - \frac{2a}{b}z + 1}\,dz \quad , \tag{209}$$

where C is the circle $|z| = 1$. The integrand $\frac{z^k}{z^2 - \frac{2a}{b}z + 1}$ has simple poles at $z_1 = \frac{a + \sqrt{a^2 - b^2}}{b}$ and $z_2 = \frac{a - \sqrt{a^2 - b^2}}{b}$, of which points only z_2 is inside the contour C. The residue of $\frac{z^k}{z^2 - \frac{2a}{b}z + 1}$ at $z = z_2$ is

$$\mathrm{Res}\left[\frac{z^k}{(z - z_1)(z - z_2)}\right]_{z=z_2} = \lim_{z\to z_2} (z - z_2)\frac{z^k}{(z - z_1)(z - z_2)}$$

$$= \frac{z_2^k}{z_2 - z_1} \quad , \tag{210}$$

which may be written

$$\mathrm{Res}\left(\frac{z^k}{z^2 - \frac{2a}{b}z + 1}\right)_{z = \frac{a-\sqrt{a^2-b^2}}{b}} = -\frac{b}{2\sqrt{a^2-b^2}}\left(\frac{a-\sqrt{a^2-b^2}}{b}\right)^k . \tag{211}$$

Hence, by Cauchy's residue theorem,

$$\int_C \frac{z^k}{z^2 - \frac{2a}{b} z + 1}\, dz = -2\pi i \frac{b}{2\sqrt{a^2 - b^2}} \left(\frac{a - \sqrt{a^2 - b^2}}{b} \right)^k . \qquad (212)$$

Now, on the circle C, $z = e^{i\varphi}$ and

$$\frac{z^k}{z^2 - \frac{2a}{b} z + 1}\, dz = i \frac{e^{ki\varphi}}{e^{i\varphi} + e^{-i\varphi} - \frac{2a}{b}}\, d\varphi$$

$$= \frac{b}{2} \frac{\sin k\varphi - i \cos k\varphi}{a - b \cos\varphi}\, d\varphi \ , \qquad (213)$$

whence it follows that

$$\int_C \frac{z^k}{z^2 - \frac{2a}{b} z + 1}\, dz = \frac{b}{2} \int_0^{2\pi} \frac{\sin k\varphi}{a - b\cos\varphi}\, d\varphi$$

$$- i \frac{b}{2} \int_0^{2\pi} \frac{\cos k\varphi}{a - b\cos\varphi}\, d\varphi \, .$$

(214)

The equality of the righthand members, and hence of the imaginary parts of the righthand members, of equations (212) and (214) leads to the conclusion

$$\int_0^{2\pi} \frac{\cos k\varphi \, d\varphi}{a - b\cos\varphi} = \frac{2\pi}{\sqrt{a^2 - b^2}} \left(\frac{a - \sqrt{a^2 - b^2}}{b} \right)^k . \qquad (215)$$

Consider now

$$I_1 = \int_{-\pi}^{\pi} [\log (a \sqrt{\cos^2 \theta + r_1^2 \sin^2 \theta}\,)] e^{-ni\theta}\, d\theta \ ,$$

(page 68) and assume that $n \neq 0$. The radicand in I_1 may be rewritten as follows:

$$\cos^2\theta + r_1^2\sin^2\theta = 1 + (r_1^2 - 1)\sin^2\theta$$
$$= 1 + \frac{r_1^2 - 1}{2}(1 - \cos 2\theta)$$
$$= \frac{r_1^2 + 1}{2} - \frac{r_1^2 - 1}{2}\cos 2\theta \quad ,$$

i.e.,

$$\cos^2\theta + r_1^2\sin^2\theta = a - b\cos 2\theta \quad , \tag{216}$$

where

$$a = \frac{r_1^2 + 1}{2} \quad \text{and} \quad b = \frac{r_1^2 - 1}{2} \quad . \tag{217}$$

It may be noted that $a > 0$ and $a^2 > b^2$. With the use of equation (216), I_1 may be written

$$I_1 = \log a \int_{-\pi}^{\pi} e^{-ni\theta}\, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} [\log (a - b\cos 2\theta)] e^{-ni\theta}\, d\theta \, , \tag{218}$$

or, since

$$\int_{-\pi}^{\pi} e^{-ni\theta}\, d\theta = 0, \text{ for } n \neq 0 \quad ,$$

$$I_1 = \frac{1}{2}\int_{-\pi}^{\pi} [\log (a - b\cos 2\theta)] e^{-ni\theta}\, d\theta \quad . \tag{219}$$

Replacement of $e^{-ni\theta}$ by $\cos n\theta - i \sin n\theta$ leads to

$$I_1 = \frac{1}{2}\int_{-\pi}^{\pi} [\log (a - b\cos 2\theta)]\cos n\theta\, d\theta - \frac{i}{2}\int_{-\pi}^{\pi} [\log (a - b\cos 2\theta)]\sin n\theta\, d\theta. \tag{220}$$

But the integrand of the second integral here is an odd

function; hence, $\int_{-\pi}^{\pi} [\log (a - b \cos 2\theta)] \sin n\theta \, d\theta = 0$, and

$$I_1 = \frac{1}{2} \int_{-\pi}^{\pi} [\log (a - b \cos 2\theta)] \cos n\theta \, d\theta \,. \qquad (221)$$

Now, integration by parts gives

$$I_1 = \frac{1}{2}\Big\{[\log (a - b \cos 2\theta)] \frac{\sin n\theta}{n} \Big\}_{-\pi}^{\pi}$$

$$- \frac{b}{n} \int_{-\pi}^{\pi} \frac{\sin n\theta \sin 2\theta}{a - b \cos 2\theta} \, d\theta$$

$$= - \frac{b}{n} \int_{-\pi}^{\pi} \frac{\sin n\theta \sin 2\theta}{a - b \cos 2\theta} \, d\theta \quad . \qquad (222)$$

Use of the identity

$$\sin n\theta \sin 2\theta = \frac{1}{2}[\cos(n - 2)\, \theta - \cos(n + 2)\, \theta] \qquad (223)$$

in equation (222) gives

$$I_1 = \frac{b}{2n} \Big\{ \int_{-\pi}^{\pi} \frac{\cos(n + 2)\theta}{a - b \cos 2\theta} \, d\theta - \int_{-\pi}^{\pi} \frac{\cos(n - 2)\theta}{a - b \cos 2\theta} \, d\theta \Big\} \quad . \qquad (224)$$

In these integrals, let $\varphi = 2\theta$, $d\varphi = 2d\theta$; then, for example,

$$\int_{-\pi}^{\pi} \frac{\cos(n + 2)\theta}{a - b \cos 2\theta} \, d\theta = 2 \int_{0}^{\pi} \frac{\cos(n + 2)\theta}{a - b \cos 2\theta} \, d\theta$$

$$= \int_{0}^{2\pi} \frac{\cos(\frac{n + 2}{2})\varphi}{a - b \cos} \, d\varphi \quad , \qquad (225)$$

and equation (224) becomes

$$I_1 = \frac{b}{2n} \Big\{ \int_{0}^{2\pi} \frac{\cos(\frac{n + 2}{2})\varphi}{a - b \cos} \, d\varphi - \int_{0}^{2\pi} \frac{\cos(\frac{n - 2}{2})\varphi}{a - b \cos} \, d\varphi \Big\} . \qquad (226)$$

Now, the values of the integrals in this equation are given

by equation (215), with $k = \frac{n+2}{2}$ and $k = \frac{n-2}{2}$, respectively. Thus,

$$I_1 = \frac{b}{2n} \cdot \frac{2\pi}{\sqrt{a^2 - b^2}} \left[\left(\frac{a - \sqrt{a^2 - b^2}}{b} \right)^{\frac{n+2}{2}} - \left(\frac{a - \sqrt{a^2 - b^2}}{b} \right)^{\frac{n-2}{2}} \right] \tag{227}$$

This expression for I_1 simplifies when it is observed from equations (217) that

$$\sqrt{a^2 - b^2} = r_1 \quad \text{and} \quad \frac{a - \sqrt{a^2 - b^2}}{b} = \frac{r_1 - 1}{r_1 + 1} = c^2 \quad , \tag{228}$$

where c is the quantity defined by equation (41). The use of these relations and equations (217) in equation (227) leads to

$$I_1 = \frac{\pi(r_1^2 - 1)(c^4 - 1)c^{n-2}}{2nr_1} \quad . \tag{229}$$

A further simplification follows from the observation that, since $c^2 = \frac{r_1 - 1}{r_1 + 1}$ and $d^2 = \frac{r_2 - 1}{r_2 + 1}$,

$$\begin{aligned} (r_1^2 - 1)(c^4 - 1) &= (r_1 + 1)(r_1 - 1)(c^2 + 1)(c^2 - 1) \\ &= - 4r_1c^2 \end{aligned}$$

and similarly, (230)

$$(r_2^2 - 1)(d^4 - 1) = - 4r_2d^2 \ .$$

With the use of these relations, equation (229) becomes, finally,

$$I_1 = - \frac{2\pi c^n}{n} \ . \tag{231}$$

Examination of the first two of equations (207) reveals that I_2 differs from I_1 only in the replacement of r_1 by r_2. It follows that, under the assumption $n \neq 0$,

$$I_2 = - \frac{2\pi d^n}{n} \quad , \tag{232}$$

where

$$d = \sqrt{\frac{r_2 - 1}{r_2 + 1}} \quad .$$

Consider next

$$I_3 = \int_{-\pi}^{\pi} [\arc \tan (\frac{r_1 \sin \theta}{\cos \theta})] e^{-ni\theta} \, d\theta$$

(page 68), and assume again that $n \neq 0$. Integration by parts, with the observation that

$$d \, [\arc \tan (\frac{r_1 \sin \theta}{\cos \theta})] = \frac{r_1 d\theta}{\cos^2 \theta + r_1^2 \sin^2 \theta} \quad , \tag{233}$$

yields

$$I_3 = \left\{ [\arc \tan (\frac{r_1 \sin \theta}{\cos \theta})] \frac{i}{n} e^{-ni\theta} \right\}_{-\pi}^{\pi}$$

$$- \frac{ir_1}{n} \int_{-\pi}^{\pi} \frac{e^{-ni\theta}}{\cos^2 \theta + r_1^2 \sin^2 \theta} \, d\theta \ ,$$

i.e.,

$$I_3 = \frac{(-1)^n \, 2\pi i}{n} - \frac{ir_1}{n} \int_{-\pi}^{\pi} \frac{\cos n\theta}{\cos^2 \theta + r_1^2 \sin \theta} \, d\theta$$

$$- \frac{r_1}{n} \int_{-\pi}^{\pi} \frac{\sin n\theta}{\cos^2 \theta + r_1^2 \sin^2 \theta} \, d\theta \ . \tag{234}$$

But $\int_{-\pi}^{\pi} \frac{\sin n\theta}{\cos^2\theta + r_1^2 \sin^2\theta}\, d\theta = 0$, since the integrand is an odd function. Hence,

$$I_3 = \frac{(-1)^n\, 2\pi i}{n} - \frac{ir_1}{n} \int_{-\pi}^{\pi} \frac{\cos n\theta}{\cos^2\theta + r_1^2 \sin^2\theta}\, d\theta \quad . \tag{235}$$

The application of equations (216), (225), (215), and (228), successively, to the integral here gives

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\cos n\theta}{\cos^2\theta + r_1^2 \sin^2\theta}\, d\theta &= \int_{-\pi}^{\pi} \frac{\cos n\theta}{a - b\cos 2\theta}\, d\theta \\ &= \int_0^{2\pi} \frac{\cos \frac{n}{2}\varphi}{a - b\cos\varphi}\, d\varphi \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \left(\frac{a - \sqrt{a^2 - b^2}}{b} \right)^{\frac{n}{2}} \\ &= \frac{2\pi c^n}{r_1} \quad , \end{aligned}$$

i.e.,

$$\int_{-\pi}^{\pi} \frac{\cos n\theta}{\cos^2\theta + r_1^2 \sin^2\theta}\, d\theta = \frac{2\pi c^n}{r_1} \quad . \tag{236}$$

The use of this result in equation (235) leads to

$$I_3 = \frac{2\pi i[(-1)^n - c^n]}{n} \tag{237}$$

Since it may be seen from equations (207) that I_4 differs from I_3 only in the replacement of r_1 by r_2, it follows, by equations (41), (42), and (237), that

$$I_4 = \frac{2\pi i[(-1)^n - d^n]}{n} \quad . \tag{238}$$

As was the case for I_3, it is assumed here that $n \neq 0$.

The integral

$$I_5 = \int_{-\pi}^{\pi} e^{-(n-2)\theta}\, d\theta$$

(page 68) can be evaluated directly to yield

$$I_5 = \begin{cases} 2\pi, & \text{for } n = 2 \\ 0, & \text{for } n = \pm 1, -2, \pm 3, \pm 4, \ldots . \end{cases} \tag{239}$$

Consider lastly

$$I_6 = \int_{-\pi}^{\pi} \theta e^{-ni\theta}\, d\theta \quad ,$$

(page 68), and assume again that $n \neq 0$. Integration by parts gives

$$I_6 = \left[\theta \cdot \frac{i}{n} e^{-ni\theta}\right]_{-\pi}^{\pi} - \frac{i}{n} \int_{-\pi}^{\pi} e^{-ni\theta}\, d\theta \quad ,$$

or

$$I_6 = \frac{(-1)^n\, 2\pi i}{n} \quad , \tag{240}$$

since

$$\int_{-\pi}^{\pi} e^{-ni\theta}\, d\theta = 0 \quad \text{for } n \neq 0.$$

Attention is returned now to c_n, given by equation (208). With I_1 through I_6 given, respectively, by equations (231), (232), and (237) through (240), equation (208) becomes, for $n = \pm 1, -2, \pm 3, \pm 4, \ldots$,

$$c_n = \frac{Ac^n}{n} + \frac{Bd^n}{n} - \frac{r_1 A[(-1)^n - c^n]}{n} - \frac{r_2 B[(-1)^n - d^n]}{n} - \frac{\rho g a^2 (-1)^n}{2n} \quad , \tag{241}$$

or, with like terms in n collected,

$$c_n = \frac{1}{n}\,[A(r_1 + 1)c^n + B(r_2 + 1)d^n$$

$$+ (-1)^{n+1}(r_1A + r_2B + \frac{\rho g a^2}{2})] \;,$$

$$n = \pm 1, -2, \pm 3, \pm 4, \ldots . \qquad (242)$$

Substitution of the values given by equations (192) for A and B in the expression $r_1A + r_2B + \frac{\rho g a^2}{2}$ shows that this expression is identically equal to zero. Hence, equation (242) may be written

$$c_n = \frac{A(r_1 + 1)c^n + B(r_2 + 1)d^n}{n}$$

$$n = \pm 1, -2, \pm 3, \pm 4, \ldots . \qquad (243)$$

For n = 2, the same equations cited for c_n with $n \neq 2$ lead to

$$c_2 = [c_n, \text{ equation } (243)]_{n=2} + \frac{\rho g a^2}{4} \;. \qquad (244)$$

The value of c_n for n = 0 will be simply denoted by c_o, not evaluated.

The constants c_n, $n = \pm 1, \pm 2, \ldots$, have been evaluated to serve in equation (205), from which the constants a_n and b_n can be evaluated. For convenience, however, the values of c_n will not be used in the following analysis until explicit expressions for a_n and b_n have been obtained.

With

$$\frac{\partial F}{\partial x} = R\{\sum_{-\infty}^{\infty} a_n \zeta_1^n + \sum_{-\infty}^{\infty} b_n \zeta_2^n\}$$

and (43)

$$\frac{\partial F}{\partial y} = R\{\sum_{-\infty}^{\infty} ir_1 a_n \zeta_1^n + \sum_{-\infty}^{\infty} ir_2 b_n \zeta_2^n\} \quad ,$$

and with $\zeta_1 = \zeta_2 = e^{i\theta}$ on the boundary of the disk, equation (205) becomes

$$R\{\sum_{-\infty}^{\infty} a_n e^{ni\theta} + \sum_{-\infty}^{\infty} b_n e^{ni\theta}\} + iR\{\sum_{-\infty}^{\infty} ir_1 a_n e^{ni\theta} + \sum_{-\infty}^{\infty} ir_2 b_n e^{ni\theta}\} = \sum_{-\infty}^{\infty} c_n e^{ni\theta} \quad . \tag{245}$$

With $\bar{a}_n$ and $\bar{b}_n$ denoting the complex conjugates of a_n and b_n, respectively, this equation may be written

$$\sum_{-\infty}^{\infty} (a_n e^{ni\theta} + \bar{a}_n e^{-ni\theta}) + \sum_{-\infty}^{\infty} (b_n e^{ni\theta} + \bar{b}_n e^{-ni\theta}) - \sum_{-\infty}^{\infty} (r_1 a_n e^{ni\theta} - r_1 \bar{a}_n e^{-ni\theta}) - \sum_{-\infty}^{\infty} (r_2 b_n e^{ni\theta} - r_2 \bar{b}_n e^{-ni\theta}) = 2\sum_{-\infty}^{\infty} c_n e^{ni\theta} \quad , \tag{246}$$

or, rearranging the terms,

$$(1 - r_1) \sum_{-\infty}^{\infty} a_n e^{ni\theta} + (1 - r_2) \sum_{-\infty}^{\infty} b_n e^{ni\theta} + (1 + r_1) \sum_{-\infty}^{\infty} \bar{a}_n e^{-ni\theta} + (1 + r_2) \sum_{-\infty}^{\infty} \bar{b}_n e^{-ni\theta} = 2\sum_{-\infty}^{\infty} c_n e^{ni\theta} \quad . \tag{247}$$

The equating now of the coefficients of $e^{ni\theta}$ in the two members of this equation gives

$$(1 - r_1)a_n + (1 + r_1)\bar{a}_{-n} + (1 - r_2)b_n + (1 + r_2)\bar{b}_{-n} = 2c_n,$$

$$n = 0, \pm 1, \pm 2, \ldots, \tag{248}$$

i.e.,

$$(1 - r_1)a_n + (1 + r_1)\bar{a}_{-n} + (1 - r_2)b_n + (1 + r_2)\bar{b}_{-n} = 2c_n$$

and (249)

$$(1 - r_1)a_{-n} + (1 + r_1)\bar{a}_n + (1 - r_2)b_{-n} + (1 + r_2)\bar{b}_n = 2c_{-n},$$

$$n = 0, 1, 2, \ldots .$$

Now, by the same argument as that given on pages 19 and 20, equations (49) and (50) apply here; i.e.,

$$a_{-n} = (-c^2)^n a_n \quad \text{and} \quad b_{-n} = (-d^2)^n b_n \quad .$$

Hence, equations (249) may be written,

$$(1 - r_1)a_n + (1 + r_1)(-c^2)^n \bar{a}_n + (1 - r_2)b_n$$

$$+ (1 + r_2)(-d^2)^n \bar{b}_n = 2c_n$$

and (250)

$$(1 - r_1)(-c^2)^n a_n + (1 + r_1)\bar{a}_n$$

$$+ (1 - r_2)(-d^2)^n b_n + (1 + r_2)\bar{b}_n = 2c_{-n} ,$$

$$n = 0, 1, 2, \ldots .$$

As has been stated earlier, the constant terms in $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, which, in equations (43), are the only terms involving a_o and b_o, have no effect on the corresponding stresses,

given by equations (6). There is no need, therefore, to evaluate a_o and b_o, and equations (250) will be considered only for the cases n = 1, 2, ... It is seen from equations (192), (243) and (244) that, for these values of n, c_n is real.

Let

$$a_n = A_n + i\alpha_n \quad \text{and} \quad b_n = B_n + i\beta_n \quad , \tag{72}$$

where A_n, α_n, B_n, and β_n are real constants. Then consideration of the real parts of equations (250), noting that c_n is real for $n \neq 0$, leads to

$$[(1 - r_1) + (1 + r_1)(-c^2)^n]A_n$$
$$+ [(1 - r_2) + (1 + r_2)(-d^2)^n]B_n = 2c_n$$

and (251)

$$[(1 - r_1)(-c^2)^n + (1 + r_1)]A_n$$
$$+ [(1 - r_2)(-d^2)^n + (1 + r_2)]B_n = 2c_{-n} ,$$
$$n = 1, 2, \ldots ;$$

and consideration of the imaginary parts leads to

$$[(1 - r_1) - (1 + r_1)(-c^2)^n]\alpha_n$$
$$+ [(1 - r_2) - (1 + r_2)(-d^2)^n]\beta_n = 0$$

and (252)

$$[(1 - r_1)(-c^2)^n - (1 + r_1)]\alpha_n$$
$$+ [(1 - r_2)(-d^2)^n - (1 + r_2)]\beta_n = 0 ,$$
$$n = 1, 2, \ldots .$$

From equations (251), now,

$$A_n = \frac{\begin{vmatrix} 2c_n & (1-r_2)+(1+r_2)(-d^2)^n \\ 2c_{-n} & (1-r_2)(-d^2)^n+(1+r_2) \end{vmatrix}}{\begin{vmatrix} (1-r_1)+(1+r_1)(-c^2)^n & (1-r_2)+(1+r_2)(-d^2)^n \\ (1-r_1)(-c^2)^n+(1+r_1) & (1-r_2)(-d^2)^n+(1+r_2) \end{vmatrix}}$$

and (253)

$$B_n = \frac{\begin{vmatrix} (1-r_1)+(1+r_1)(-c^2)^n & 2c_n \\ (1-r_1)(-c^2)^n+(1+r_1) & 2c_{-n} \end{vmatrix}}{\begin{vmatrix} (1-r_1)+(1+r_1)(-c^2)^n & (1-r_2)+(1+r_2)(-d^2)^n \\ (1-r_1)(-c^2)^n+(1+r_1) & (1-r_2)(-d^2)^n+(1+r_2) \end{vmatrix}}$$

Evaluation of the determinants and subsequent simplification yield

$$A_n = -\frac{r_2(c_n+c_{-n})[1-(-d^2)^n]+(c_n-c_{-n})[1+(-d^2)^n]}{D_n}$$

and (254)

$$B_n = \frac{r_1(c_n+c_{-n})[1-(-c^2)^n]+(c_n-c_{-n})[1+(-c^2)^n]}{D_n} ,$$

where

$$D_n = (r_1-r_2)[1-(c^2d^2)^n]-(r_1+r_2)[(-c^2)^n-(-d^2)^n] ,$$

$$n = 1, 2, \ldots . \qquad (97)$$

It was shown in Chapter II, pages 28 through 33, that $D_n \neq 0$ for $n = 1, 2, \ldots$. From equation (243), which gives the values of c_n for $n = \pm 1, -2, \pm 3, \pm 4, \ldots$, it follows that

$$c_n + c_{-n} = \frac{1}{n}[A(r_1 + 1)(c^n - c^{-n}) + B(r_2 + 1)(d^n - d^{-n})] \qquad (255)$$

and

$$c_n - c_{-n} = \frac{1}{n}[A(r_1 + 1)(c^n + c^{-n}) + B(r_2 + 1)(d^n + d^{-n})] , \quad n = 1, 3, 4, \ldots \qquad (256)$$

Use of these equations in equations (254), followed by some straightforward manipulation, yields the results

$$A_n = -\frac{1}{nD_n}\{(r_2 + 1)[\frac{A(r_1 + 1)(c^{2n} + (-d^2)^n)}{c^n} + (1 + (-1)^n)B(r_2 + 1)d^n] - (r_2 - 1)[\frac{A(r_1 + 1)(1 + (-c^2d^2)^n)}{c^n} + \frac{B(r_2 + 1)(1 + (-d^4)^n}{d^n}]\} \qquad (257)$$

and

$$B_n = \frac{1}{nD_n}\{(r_1 + 1)[(1 + (-1)^n)A(r_1 + 1)c^n + \frac{B(r_2 + 1)((-c^2)^n + d^{2n})}{d^n}]$$

$$- (r_1 - 1)\left[\frac{A(r_1 + 1)(1 + (-c^4)^n)}{c^n}\right.$$

$$\left.+ \frac{B(r_2 + 1)(1 + (-c^2d^2)^n)}{d^n}\right]\Big\} \quad ,$$

$$n = 1, 3, 4, \ldots \quad (258)$$

From equation (244), which gives the value of c_2, it follows that

$$c_2 + c_{-2} = [c_n + c_{-n}, \text{ equation (255)}]_{n=2} + \frac{\rho g a^2}{4}$$

and (259)

$$c_2 - c_{-2} = [c_n - c_{-n}, \text{ equation (256)}]_{n=2} + \frac{\rho g a^2}{4} \ .$$

Use of these equations in equations (254) leads to

$$A_2 = [A_n, \text{ equation (257)}]_{n=2} - \frac{\rho g a^2[r_2 + 1 - (r_2 - 1)d^4]}{4D_2}$$

and (260)

$$B_2 = [B_n, \text{ equation (258)}]_{n=2} + \frac{\rho g a^2[r_1 + 1 - (r_1 - 1)c^4]}{4D_2}$$

Consider next equations (252) in α_n and β_n. The determinant of these homogeneous equations, considered simultaneously, is

$$\Delta_n = \begin{vmatrix} (1-r_1)-(1+r_1)(-c^2)^n & (1-r_2)-(1+r_2)(-d^2)^n \\ (1-r_1)(-c^2)^n-(1+r_1) & (1-r_2)(-d^2)^n-(1+r_2) \end{vmatrix} \quad (261)$$

Upon addition of the second row to the first, removal of the

factor 2 from the new first row, and subtraction of the resulting first row from the second, it is seen that

$$\Delta_n \equiv 2D'_n \ , \tag{262}$$

where D'_n is the determinant given by the second of equations (77). It was shown in Chapter II, pages 28 through 33, that $D'_n \neq 0$ for $n = 2, 3, \ldots$, and, as stated on page 33, calculation shows that $D'_1 = 0$. Hence,

$$\Delta_1 = 0 \ ,$$

but (263)

$$\Delta_n \neq 0 \ , n = 2, 3, \ldots .$$

It follows from this last statement and equations (252) that

$$\alpha_n = \beta_n = 0, \; n = 2, 3, \ldots , \tag{264}$$

but that, from the second of equations (252) and equations (41) and (42),

$$\begin{aligned}
\alpha_1 &= -\frac{(1 - r_2)(-d^2) - (1 + r_2)}{(1 - r_1)(-c^2) - (1 + r_1)}\beta_1 \\
&= -\frac{1 + r_1}{1 + r_2}\,\frac{(1 - r_2)^2 - (1 + r_2)^2}{(1 - r_1)^2 - (1 + r_1)^2}\beta_1 \\
&= -\frac{r_2(r_1 + 1)}{r_1(r_2 + 1)}\beta_1 \ ,
\end{aligned} \tag{265}$$

where the constant β_1 is arbitrary. But this relation between α_1 and β_1, with β_1 arbitrary, is the same as equation (101), Chapter II. Since the same stress function (except

for constants) as that of Chapter II is being used here, it follows that the contributions to the stresses here of terms involving α_1 and β_1 are the same as the contributions of the terms involving α_1 and β_1 to the stresses in the problem of Chapter II, which contributions were shown on pages 32 and 35 to be zero. It is convenient to set (as in Chapter II)

$$\alpha_1 = \beta_1 = 0 \quad . \tag{102}$$

For convenience, the formulas for the values of a_n and b_n, $n = 1, 2, \ldots$, obtained from equations (72), (257), (258), (260), (264), and (102), are given below.

$$a_n = -\frac{1}{nD_n}\{(r_2+1)[\frac{A(r_1+1)(c^{2n}+(-d^2)^n)}{c^n} + (1+(-1)^n)B(r_2+1)d^n]$$
$$- (r_2-1)[\frac{A(r_1+1)(1+(-c^2d^2)^n)}{c^n} + \frac{B(r_2+1)(1+(-d^4)^n)}{d^n}]\},$$
$$n = 1, 3, 4, \ldots, \tag{266}$$

$$b_n = \frac{1}{nD_n}\{(r_1+1)[(1+(-1)^n)A(r_1+1)c^n + \frac{B(r_2+1)((-c^2)^n+d^{2n})}{d^n}]$$
$$- (r_1-1)[\frac{A(r_1+1)(1+(-c^4)^n)}{c^n} + \frac{B(r_2+1)(1+(-c^2d^2)^n)}{d^n}]\},$$
$$n = 1, 3, 4, \ldots, \tag{267}$$

$$a_2 = [a_n, \text{ equation } (266)]_{n=2} - \frac{\rho g a^2[r_2+1-(r_2-1)d^4]}{4D_2} \quad , \tag{268}$$

and

$$b_2 = [b_n, \text{ equation } (267)]_{n=2} + \frac{\rho g a^2[r_1+1-(r_1-1)c^4]}{4D_2} \quad , \tag{269}$$

where

$$D_n = (r_1-r_2)[1-(c^2d^2)^n] - (r_1+r_2)[(-c^2)^n-(-d^2)^n] \quad , \qquad n = 1, 2, \ldots . \tag{97}$$

With these constants evaluated, the solution of the problem is essentially complete.

As stated at the outset of this chapter, the actual stresses in the disk considered here are obtainable as the resultants of the stresses due to the supporting force, given by equations (192) and (193), and the stresses due to the weight of the disk, given by equations (110) through (112) and equations (266) through (269). Thus, insertion of the values of a_n and b_n given by equations (266) through (269) into equation (110) and addition of the result to the first of equations (193) lead to

$$\sigma_x = R\{\sum_{n=1}^{\infty} [A(r_1+1)\frac{(r_2+1)(c^{2n}+(-d^2)^n)-(r_2-1)(1+(-c^2d^2)^n)}{c^n}$$

$$+ B(r_2+1)\frac{(r_2+1)(1+(-1)^n)d^{2n}-(r_2-1)(1+(-d^4)^n)}{d^n}]$$

$$\frac{r_1^2[\varsigma_1^n-(-c^2)^n\varsigma_1^{-n}]}{W_1D_n}$$

$$- \sum_{n=1}^{\infty} [A(r_1+1)\frac{(r_1+1)(1+(-1)^n)c^{2n}-(r_1-1)(1+(-c^4)^n)}{c^n}$$

$$+ B(r_2+1)\frac{(r_1+1)((-c^2)^n+d^{2n})-(r_1-1)(1+(-c^2d^2)^n)}{d^n}]$$

$$\frac{r_2^2[\varsigma_2^n-(-d^2)^n\varsigma_2^{-n}]}{W_2D_n}$$

$$+ \rho g a^2[r_2+1-(r_2-1)d^4]\frac{r_1^2(\zeta_1^2-c^4\zeta_1^{-2})}{2W_1D_2}$$

$$- \rho g a^2[r_1+1-(r_1-1)c^4]\frac{r_2^2(\zeta_2^2-d^4\zeta_2^{-2})}{2W_2D_2}\}$$

$$- \frac{r_1^2 Ax}{x^2+r_1^2y^2} - \frac{r_2^2 Bx}{x^2+r_2^2y^2} - \rho g x \ , \tag{270}$$

where

$$A = -\frac{\rho g a^2 r_1(1+r_2^2\nu_{yx})}{2(r_1^2-r_2^2)} \quad \text{and} \quad B = \frac{\rho g a^2 r_2(1+r_1^2\nu_{yx})}{2(r_1^2-r_2^2)} \ . \tag{192}$$

Similarly, with equations (111) and the second of equations (193) playing the roles of equations (110) and the first of equations (193), respectively,

$$\sigma_y = R\{-\sum_{n=1}^{\infty}[A(r_1+1)\frac{(r_2+1)(c^{2n}+(-d^2)^n)-(r_2-1)(1+(-c^2d^2)^n)}{c^n}$$

$$+ B(r_2+1)\frac{(r_2+1)(1+(-1)^n)d^{2n}-(r_2-1)(1+(-d^4)^n)}{d^n}]$$

$$\frac{\zeta_1^n-(-c^2)^n\zeta_1^{-n}}{W_1D_n}$$

$$+\sum_{n=1}^{\infty}[A(r_1+1)\frac{(r_1+1)(1+(-1)^n)c^{2n}-(r_1-1)(1+(-c^4)^n)}{c^n}$$

$$+ B(r_2+1)\frac{(r_1+1)((-c^2)^n+d^{2n})-(r_1-1)(1+(-c^2d^2)^n)}{d^n}]$$

$$\frac{\zeta_2^n-(-d^2)^n\zeta_2^{-n}}{W_2D_n}$$

$$- \rho g a^2[r_2+1-(r_2-1)d^4]\frac{\zeta_1^2-c^4\zeta_1^{-2}}{2W_1D_2}$$

$$+ \rho g a^2[r_1+1-(r_1-1)c^4]\frac{\zeta_2^2-d^4\zeta_2^{-2}}{2W_2D_2}\}$$

$$+ \frac{Ax}{x^2+r_1^2y^2} + \frac{Bx}{x^2+r_2^2y^2} - \rho g x \quad , \tag{271}$$

and, with equations (112) and the third of equations (193) playing these respective roles,

$$\tau_{xy} = R\{i\sum_{n=1}^{\infty}[A(r_1+1)\frac{(r_2+1)(c^{2n}+(-d^2)^n)-(r_2-1)(1+(-c^2d^2)^n)}{c^n}$$

$$+ B(r_2+1)\frac{(r_2+1)(1+(-1)^n)d^{2n}-(r_2-1)(1+(-d^4)^n)}{d^n}]$$

$$\frac{r_1[\zeta_1^n-(-c^2)^n\zeta_1^{-n}]}{W_1D_n}$$

$$- i\sum_{n=1}^{\infty}[A(r_1+1)\frac{(r_1+1)(1+(-1)^n)c^{2n}-(r_1-1)(1+(-c^4)^n)}{c^n}$$

$$+ B(r_2+1)\frac{(r_1+1)((-c^2)^n+d^{2n})-(r_1-1)(1+(-c^2d^2)^n)}{d^n}]$$

$$\frac{r_2[\zeta_2^n-(-d^2)^n\zeta_2^{-n}]}{W_2D_n}$$

$$+ \rho g a i[r_2+1-(r_2-1)d^4]\frac{r_1(\zeta_1^2-c^4\zeta_1^{-2})}{2W_1D_2}$$

$$- \rho g a i[r_1+1-(r_1-1)c^4]\frac{r_2(\zeta_2^2-d^4\zeta_2^{-2})}{2W_2D_2}\}$$

$$- \frac{r_1^2 Ay}{x^2+r_1^2y^2} - \frac{r_2^2 By}{x^2+r_2^2y^2} \;, \tag{272}$$

where, as for σ_x, A and B are given by equations (192). The similarities of these stress formulas permit the writing of the composite formula

$$\begin{aligned}
S_k = R\Big\{ & T_k \sum_{n=1}^{\infty} \Big[A(r_1+1) \frac{(r_2+1)(c^{2n}+(-d^2)^n)-(r_2-1)(1+(-c^2d^2)^n)}{c^n} \\
& + B(r_2+1) \frac{(r_2+1)(1+(-1)^n)d^{2n}-(r_2-1)(1+(-d^4)^n)}{d^n} \Big] \\
& \qquad \frac{\zeta_1^n-(-c^2)^n\zeta_1^{-n}}{W_1 D_n} \\
& + U_k \sum_{n=1}^{\infty} \Big[A(r_1+1) \frac{(r_1+1)(1+(-1)^n)c^{2n}-(r_1-1)(1+(-c^4)^n)}{c^n} \\
& + B(r_2+1) \frac{(r_1+1)((-c^2)^n+d^{2n})-(r_1-1)(1+(-c^2d^2)^n)}{d^n} \Big] \\
& \qquad \frac{\zeta_2^n-(-d^2)^n\zeta_2^{-n}}{W_2 D_n} \\
& + T_k \; \frac{\rho g a^2[r_2+1-(r_2-1)d^4][\zeta_1^2-c^4\zeta_1^{-2}]}{2W_1D_2} \\
& + U_k \; \frac{\rho g a^2[r_1+1-(r_1-1)c^4][\zeta_2^2-d^4\zeta_2^{-2}]}{2W_2D_2} \Big\} \\
& - V_k \, \rho g x + X_k \frac{A}{x^2+r_1^2y^2} + Y_k \frac{B}{x^2+r_2^2y^2} \;, \quad k = 1,\ 2,\ 3,
\end{aligned} \tag{273}$$

where S_k, T_k, U_k, V_k, X_k, and Y_k are given by the table

below.

k	S_k	T_k	U_k	V_k	X_k	Y_k
1	σ_x	r_1^2	$-r_2^2$	1	$-r_1^2 x$	$-r_2^2 x$
2	σ_y	-1	1	1	x	x
3	τ_{xy}	ir_1	$-ir_2$	0	$-r_1^2 y$	$-r_2^2 y$

3.4. Displacements.

Those components of the displacements resulting from the stresses due to the weight of the disk have been obtained, except for the new values of the constants, in section 2.5 and are given by equations (139) and (140), together with equations (266) through (269) for the values of a_n and b_n, $n = 1, 2, \ldots$. To find the contributions, call them u' and v', of the stresses due to the supporting force at the center of the disk, let the values of the stresses σ_x' and σ_y' given by the first two of equations (193) be substituted into the first two of equations (2) to yield, after simplification,

$$\frac{\partial u'}{\partial x} = -\frac{r_1^2+\nu_{xy}}{E_x}\,\frac{Ax}{x^2+r_1^2y^2} - \frac{r_2^2+\nu_{xy}}{E_x}\,\frac{Bx}{x^2+r_2^2y^2}$$

and (274)

$$\frac{\partial v'}{\partial y} = \frac{r_1^2\nu_{yx}+1}{E_y}\,\frac{Ax}{x^2+r_1^2y^2} + \frac{r_2^2\nu_{yx}+1}{E_y}\,\frac{Bx}{x^2+r_2^2y^2} \quad .$$

Equation (9) has been used here to simplify the coefficients. Integration of the first of equations (274) with respect to

x and the second with respect to y gives

$$u' = -\frac{r_1^2+\nu_{xy}}{E_x}\frac{A}{2}\log(x^2+r_1^2y^2) - \frac{r_2^2+\nu_{xy}}{E_x}\frac{B}{2}\log(x^2+r_2^2y^2)+f(y)$$

and (275)

$$v' = \frac{r_1^2\nu_{yx}+1}{E_y}\frac{A}{r_1}\text{Arc tan}\frac{r_1y}{x} + \frac{r_2^2\nu_{yx}+1}{E_y}\frac{B}{r_2}\text{Arc tan}\frac{r_2y}{x}+g(x) ,$$

where f(y) and g(x) are as yet undetermined functions of their arguments. These functions will be determined by the method used in section 2.5.

The derivatives of u' and v' with respect to y and x, respectively, obtained from the equations above, are

$$\frac{\partial u'}{\partial y} = -\frac{r_1^2+\nu_{xy}}{E_x}\frac{r_1^2Ay}{x^2+r_1^2y^2} - \frac{r_2^2+\nu_{xy}}{E_x}\frac{r_2^2By}{x^2+r_2^2y^2} + \frac{df(y)}{dy}$$

and (276)

$$\frac{\partial v'}{\partial x} = -\frac{r_1^2\nu_{yx}+1}{E_y}\frac{Ay}{x^2+r_1^2y^2} - \frac{r_2^2\nu_{yx}+1}{E_y}\frac{By}{x^2+r_2^2y^2} + \frac{dg(x)}{dx} .$$

With these expressions for the derivatives, the last of equations (2), with τ_{xy} replaced by τ'_{xy}, yields

$$\tau'_{xy} = -G_{xy}\Big[\Big(\frac{r_1^2+\nu_{xy}}{E_x} + \frac{r_1^2\nu_{yx}+1}{r_1^2E_y}\Big)\frac{r_1^2Ay}{x^2+r_1^2y^2}$$

$$+ \Big(\frac{r_2^2+\nu_{xy}}{E_x} + \frac{r_2^2\nu_{yx}+1}{r_2^2E_y}\Big)\frac{r_2^2By}{x^2+r_2^2y^2} - \frac{df(y)}{dy} - \frac{dg(x)}{dx}\Big] .$$

(277)

But, by equations (130), the quantities in parentheses are each equal to $\frac{1}{G_{xy}}$. Hence, the equation above may be written

$$\tau'_{xy} = - \frac{r_1^2 Ay}{x^2+r_1^2 y^2} - \frac{r_2^2 By}{x^2+r_2^2 y^2} + G_{xy}\left[\frac{df(y)}{dy} + \frac{dg(x)}{dx}\right] . \tag{278}$$

Comparison of this equation with the third of equations (193) indicates that

$$G_{xy}\left[\frac{df(y)}{dy} + \frac{dg(x)}{dx}\right] = 0 \quad , \tag{279}$$

i.e., since $G_{xy} \neq 0$,

$$\frac{df(y)}{dy} = - \frac{dg(x)}{dx} \quad . \tag{280}$$

Since the lefthand member of this equation is a function of y only, while the righthand member is a function of x only, the fact that the equation must hold for all points (x, y) within the disk leads to the conclusion that $\frac{df(y)}{dy} = - \frac{dg(x)}{dx} = k$, where k is some constant. Integration of these functions gives

$$f(y) = ky + m$$

and (281)

$$g(x) = - kx + n,$$

where m and n are arbitrary constants. These constants m and n represent constant contributions to u' and v', as is seen from equations (275), and hence represent a rigid displacement of the whole disk; it follows that, if the support

is to be stationary, then m = n = 0. Also, the fact that the displacement u' (in the x-direction) is symmetric with respect to the x-axis requires, considering the first of equations (275), that f(y) be an even function, i.e., that k = 0. With k = m = n = 0,

$$f(y) \equiv g(x) \equiv 0 \quad , \tag{282}$$

and equations (275) become

$$u' = -\frac{A(r_1^2 + \nu_{xy})}{2E_x}\log(x^2 + r_1^2y^2) - \frac{B(r_2^2 + \nu_{xy})}{2E_x}\log(x^2 + r_2^2y^2) \tag{283}$$

and

$$v' = \frac{A(r_1^2\nu_{yx} + 1)}{r_1E_y}\operatorname{Arc\,tan}\frac{r_1y}{x} + \frac{B(r_2^2\nu_{yx} + 1)}{r_2E_y}\operatorname{Arc\,tan}\frac{r_2y}{x} \quad . \tag{284}$$

Now the actual displacements in the disk are given by the sum of equations (139) and (283) and the sum of equations (140) and (284); thus,

$$u = -R\{\sum_{n=1}^{\infty}[\frac{r_1^2+\nu_{xy}}{E_x}a_n(\zeta_1^n+(-c^2)^n\zeta_1^{-n})+\frac{r_2^2+\nu_{xy}}{E_x}b_n(\zeta_2^n+(-d^2)^n\zeta_2^{-n})]\}$$
$$-\frac{\rho\varepsilon}{2}(\frac{1-\nu_{xy}}{E_x}x^2 + \frac{\nu_{yx}+1}{E_y}y^2)$$
$$-\frac{A(r_1^2+\nu_{xy})}{2E_x}\log(x^2+r_1^2y^2) - \frac{B(r_2^2+\nu_{xy})}{2E_x}\log(x^2+r_2^2y^2) \tag{285}$$

and

$$v = -R\{i\sum_{n=1}^{\infty}[\frac{r_1^2\nu_{yx}+1}{r_1E_y}a_n(\varsigma_1^n+(-c^2)^n\varsigma_1^{-n})+\frac{r_2^2\nu_{yx}+1}{r_2E_y}b_n(\varsigma_2^n+(-d^2)^n\varsigma_2^{-n})]\}$$

$$+\frac{\nu_{yx}-1}{E_y}\rho gxy+\frac{A(r_1^2\nu_{yx}+1)}{r_1E_y}\text{Arc tan}\frac{r_1y}{x}$$

$$+\frac{B(r_2^2\nu_{yx}+1)}{r_2E_y}\text{Arc tan}\frac{r_2y}{x}\quad, \tag{286}$$

with A and B given by equations (192) and a_n and b_n given by equations (266) through (269).

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BIOGRAPHICAL SKETCH

James Blake Wilson was born in Albion, Michigan, on February 9, 1924. In 1927 his family moved to Gainesville, Florida, where he received his elementary, high school, and undergraduate college education. The author's study at the University of Florida was interrupted by service in the Army from June, 1943, to May, 1946. He received the Bachelor of Science degree in mathematics, with honors, from the University in February, 1948, and the Master of Science Degree in mathematics and mechanics at Cornell University in June, 1951. At Cornell he held the Erastus Brooks Fellowship in Mathematics during his first year and instructed parttime in mathematics for five semesters and mechanics for one semester.

The author was an instructor in Ordnance Engineering at the United States Military Academy from August, 1951, to August, 1954, and, a captain in the Army Reserve, he is designated to return to that post in the event of mobilization.

In September, 1954, the author entered the Graduate School of the University of Florida, where he was a graduate assistant in mathematics for three semesters and a research assistant for seven months. He received a graduate fellowship for his final year of study.

The author is a member of Phi Beta Kappa and Phi Kappa Phi honor societies and of the American Mathematical Society and the Mathematical Association of America.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June 3, 1957

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